MCAMC 2019

MCAMC Sample: Team Round

Do not turn the page until you are told to do so.

This section of the competition is to be completed **in your team** within **1 hour**, and this sections consists of **20 questions**. No aids such as calculators, notes, compasses, etc. are allowed. All answers must be recorded on this page in order to receive credit. Answers must be exact (do not approximate π) and in simplest form, with all fractions expressed as improper fractions. Examples of unacceptable answers include: $\frac{4}{6}$, $1\frac{1}{3}$, $3+2$. Examples of acceptable answers include $\frac{2}{3}$, $\frac{4}{3}$, 5.

Team Name: ______________________ Team ID: ____________

1.	________	6.	________	11.	________	16.	________
2.	________	7.	________	12.	________	17.	________
3.	________	8.	________	13.	________	18.	________
4.	________	9.	________	14.	________	19.	________
5.	________	10.	________	15.	________	20.	________

NOTE: In this sample, only 10 questions across 2 sections are provided. The real competition will include 20 questions across more sections. The real competition will use a different topic than the one provided here, and the topic will NOT be one that is traditionally covered in middle school and high school math courses, so that no student will have an advantage coming in (contrary to the topic presented here). Like the topic presented here, the only background knowledge expected will be familiarity with variables and equations. Questions will not necessarily be in difficulty order in this section.

Introduction: Log Me!

The topic of today's team round is *logarithms* (often called logs). Logs are a fundamental tool of mathematics, finding applications just about everywhere: computer science, chemistry, physics, statistics, and much, much more. Today, we will develop the mathematical fundamentals of logs and explore a few related brain teasers.

A log, in short, is the opposite of an exponent. The following definition makes this precise:

Definition. The *logarithm* of a positive number b with respect to positive base a is the number x that satisfies $a^x = b$. In other words, the statements $a^x = b$ and $x = \log_a b$ (read "log base a of b") are equivalent.

Note that logs are undefined when $b = 0$ or when the base $a = 1$. (Think about why!)

Problem 1. If $3^6 = 729$, find $\log_3 729$.

Problem 2. What is the value of $\log_{\sqrt{2}} 128$?

Problem 3. If a, b, and c are whole numbers between 1 and 10, inclusive, then how many ways can you choose a, b, and c such that they satisfy $\log_a b = c$?

Problem 4. To determine the age of a fossil, we can use the formula $t = 5730 \log_2 \frac{N_0}{N}$, where t is the age in years, N_0 is the original number of Carbon-14 atoms, and N is the number of remaining Carbon-14 atoms. If a fossil originally had 480 Carbon-14 atoms and now has 30 Carbon-14 atoms, how old is the fossil (in years)?

Properties of Logarithms

Like most other operations, logarithms have interesting properties that we can pay special attention to. They will help us solve complicated problems more easily.

Properties of Logarithms.

(i) $\log_a b + \log_a c = \log_a (b \times c)$

(ii) See Problem 5.

(iii) $\log_a b = \dfrac{1}{\log_b a}$

(iv) $\log_a b = \dfrac{\log_c b}{\log_c a}$ (for any c of your choice)

Proof.

(i) Let $x = \log_a b$ and $y = \log_a c$. By the definition, this is equivalent to $a^x = b$ and $a^y = c$. Because $a^x = b$ and $a^y = c$, we can multiply them to get $a^x \times a^y = b \times c$. Using exponent laws, this becomes $a^{x+y} = b \times c$. Now, using the definition again, this is equivalent to $x + y = \log_a(b \times c)$. Plugging back in x and y, we have $\log_a b + \log_a c = \log_a(b \times c)$, which is what we wanted to show.

(ii) See Problem 5.

(iii) (This proof is provided for your interest and is not important to the problems.) Let $x = \log_a b$ and $y = \log_b a$. By the definition, this is equivalent to $a^x = b$ and $b^y = a$. Because $b^y = a$, we can write $b^{xy} = a^x$, and from before, $a^x = b$, so $b^{xy} = b$. The two sides can only be equal when the exponents are equal, that is, $xy = 1$. Rearranging we get $x = \frac{1}{y}$, and plugging back in x and y, we get what we wanted to show.

(iv) The proof of this property is slightly beyond the scope of this team round. □

Properties (iii) and (iv) are especially useful when needing to simplify an expression with logarithms of many different bases. One common strategy is to find a common base using properties (iii) and (iv), then combine with properties (i) and (ii).

Problem 5. Complete the right side of the equation: $\log_a b - \log_a c =?$. Your answer should include a single log. (Hint: you can do something similar to the proof of property (i).)

Problem 6. What is the value of $(\log_6 2 + \log_6 18)(\log_{25} 125 - \log_{25} 5)$?

Problem 7. What is the value of $\log_{24} 1 + \log_{24} 2 + \dfrac{1}{\log_3 24} + \dfrac{\log_2 4}{\log_2 24}$?

Problem 8. Given that $a, b > 1$, when does $\log_a b = \log_b a$? Your answer should be "never", "always", or a description or list of value(s) for a and b that satisfy the equation.

Problem 9. Given that $a, b, c, d > 1$, when does $\log_a bc + \log_a d = \log_a b + \log_a cd$? Your answer should be "never", "always", or a description or list of value(s) for a, b, c, and d that satisfy the equation.

Problem 10. Find the value of a that satisfies $\log_{10}(a + a) = 2 \log_{10} a$.