MCAMC

# Team Round

Middlesex County Academy Math Competition

April 12, 2025

This section of the competition consists of **15 questions** to be completed by **your team** within **1 hour**.

**No** calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

Answers must be **exact** (do not approximate $\pi$) and in **simplest form**, with all fractions expressed as improper fractions.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Some questions will require a brief explanation. Additionally, questions may have no answer. If so, the correct response is **"None"**. Problems that are not proof-based do not need an explanation.

A note on proofs: The proofs in this team round are designed so that **you will not need more than** 5 **or** 6 **sentences**. A good explanation should be brief, to the point, and easy to follow.

**Best of luck!**

# 1 Answer Sheet

Names: ____________________

Individual IDs: ____________________

Team Name: ____________________

Team ID: ____________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

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# 2 Prologue

This round provides a thorough exposition for **Linear Algebra**, one of the most important subjects in mathematics.

What exactly is Linear Algebra? Let's start with the second word. When we think of algebra, we think of equations and variables - solving for x, or y, or maybe both. Sometimes equations can be simple looking, such as

$$x + 1 = 3,$$

or they can be more complicated, such as

$$x^4 + x^5 = e^6.$$

The first word defines more clearly what we're talking about. Linear Algebra is the study of linear equations, which are of the form

$$a_1x_1 + \cdots + a_kx_k = C$$

for constants $a_1, \ldots, a_k, C$ and variables $x_1, \ldots, x_k$.

Solving sets of linear equations is often extremely important in different fields. One such example is one of our favorite search engines: Google! The reason why Google can return relevant results in response to a query is in part because of the **PageRank Algorithm**, a linear-algebra based algorithm that ranks web pages based on how many times they are referenced by other web pages.

This Team Round scrapes only the top of Linear Algebra - there is so much to the subject that is impossible to cover in 60 minutes. **In fact, you won't even be solving linear equations for most of this round!** Instead, this round is designed to give you a solid foundation of Linear Algebra principles, and give you a glimpse of the capabilities it offers.

# 3 Preliminaries

One of the most important considerations in Linear Algebra is: what type of space are we dealing with? You are likely familiar with a few such spaces.

The most simple case, 1-dimensional space, such as a number line, points are defined by a single number: $x$.

In 2-dimensional space, the **Cartesian Plane** is used to describe points using $x$ and $y$ axes - such points are described by two numbers, in coordinate form $(x, y)$.

You may even have heard of 3-dimensional spaces, which contain figures like spheres and rectangular prisms. In this case, points are defined by $(x, y, z)$.

However, dimensions don't stop at 3. Let's generalize this notion.

**Definition.** We say that $n$-dimensional space, denoted $\mathbb{R}^n$, is the set of all coordinates $(a_1, \ldots, a_n) \in \mathbb{R}$. ($\in$ means in, and $\mathbb{R}$ is the set of all real numbers.)

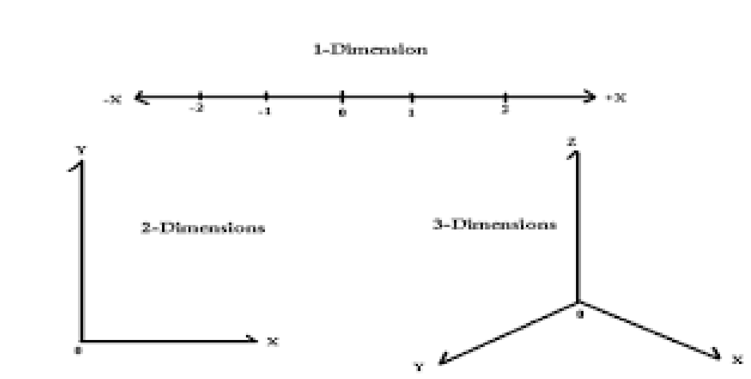


Figure 1: 3 familiar coordinate systems. Also known as $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$.

**Example 1.** Consider the three points $A, O, B$ on the Cartesian Plane in Figure 2. The point $O$, known as the **origin**, is represented by the pair of numbers $(0, 0)$. The point $A$ is represented by $(4, 3)$.

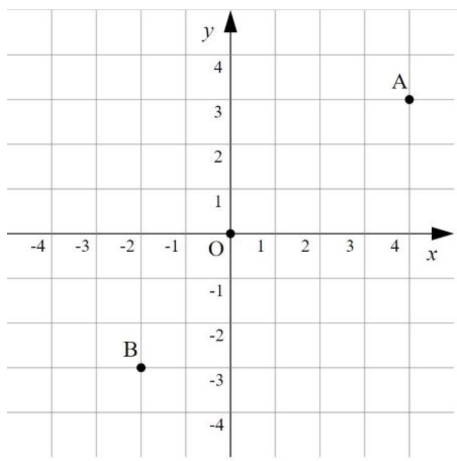


Figure 2: Points on the Cartesian Plane.

**Problem 1.** What is point $B$ in Figure 2?

**Problem 2.** What is the distance $OA$ in Figure 2? What about the distance $OB$?

# 4 Vectors

**Definition.** A vector $\mathbb{R}^n$ is an ordered list of numbers, called **components**, and is written in the form of $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, where $a_1, \ldots, a_n \in \mathbb{R}$. Vectors are denoted with a bolded letter, oftentimes $\mathbf{v}$.

Vectors are endowed with two intrinsic attributes: **magnitude** and **direction**. The magnitude (sometimes called norm) of a vector is it's length in units. vector using double brackets: $||\mathbf{v}||$. For a vector with components $a_1, \ldots, a_n$ in that order, this magnitude is equal to

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

For example, in Figure 3, $||\mathbf{v}|| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

Each vector also carries with it a direction; the best way to understand this is to think about a vector as an arrow, from the origin to the point $(a_1, a_2, \ldots, a_n)$. Each vector has a **tail** (where the arrow starts) and **head** (here the arrow ends).

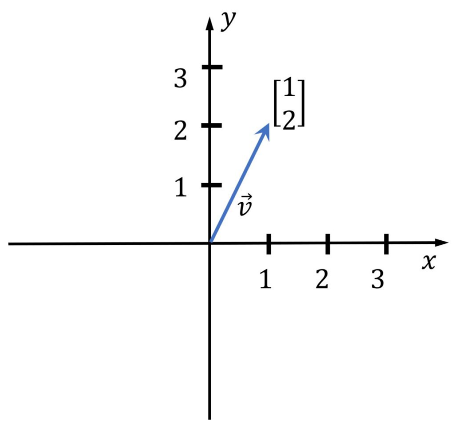


Figure 3: The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

An important note about vectors is that they do not have a fixed starting and ending point; any **translation** (i.e. moving in some direction) of a vector results in the same vector itself. So, for example, the vector is tail $(1, 1)$ and head $(3, 2)$ is also the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

What can do with vectors? Let's define some basic operations that can be done on them.

**Definition.** *(Scalar Multiplication.)* Let $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$. Then, $c\mathbf{v}$ is the result of multiplying every component of $\mathbf{v}$ by $c$. For example, $2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$. We say that $c$ is a **scalar**.

**Definition.** *(Vector Addition and Subtraction.)* Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{u} + \mathbf{v}$ is the vector resulting from adding $\mathbf{u}$ and $\mathbf{v}$ componentwise, and $\mathbf{u} - \mathbf{v}$ is the vector resulting fron subtracting $\mathbf{v}$ from $\mathbf{u}$ componentwise. For example, $\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$.

**Example 2.** Set $\mathbf{v} \in \mathbb{R}^n$ and let $b, c$ be scalars. Then, $b(c\mathbf{v}) = (bc)\mathbf{v}$.

*Proof.* Write $\mathbf{v} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$. Then, note that

$$b(c\mathbf{v}) = b\left(c\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}\right) = b\begin{bmatrix}ca_1\\ca_2\\\vdots\\ca_n\end{bmatrix} = \begin{bmatrix}bca_1\\bca_2\\\vdots\\bca_n\end{bmatrix} = bc\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = (bc)\mathbf{v}.$$

$\square$

Now, we have enough to establish some properties of vectors. You'll notice some similarities with the set of real numbers and operations we are already familiar with.

**Properties of Vectors.** Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$.

1. Associativity of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. Commutativity of Addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. Additive Identity: There exists a vector $\mathbf{0} \in \mathbb{R}^n$, the **zero vector**, with $0 + \mathbf{v} = \mathbf{v} + 0 = \mathbf{v}$.
4. Additive Inverse: There exists an element $-\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. Distributivity of Vector Addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
6. Distributivity of Scalar Addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

**Problem 3.** Prove Associativity of Addition.

**Problem 4.** Prove Distributivity of Scalar Addition.

**Problem 5.** Show that, for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the inequality $||\mathbf{u}|| + ||\mathbf{v}|| \geqslant ||\mathbf{u} + \mathbf{v}||$ always holds. What is the equality case?

# 5 Dot Products

Associated with vectors is a special type of product, known as the **dot product**. First, let's see how to calculate it.

**Definition.** *(Dot Product.)* The dot product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with components

$$\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**Problem 6.** Prove that $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$ for all vectors $\mathbf{v} \in \mathbb{R}^n$.

The dot product has two important properties, similar to those of vectors; it is both commutative, meaning that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, and distributive, meaning that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

One of the biggest properties of dot products is revealed geometrically. Consider two vectors, $\mathbf{u}$ and $\mathbf{v}$, in $\mathbb{R}^n$. Note that $\mathbf{u} + \mathbf{v}$ is the vector from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$, as in Figure **??**

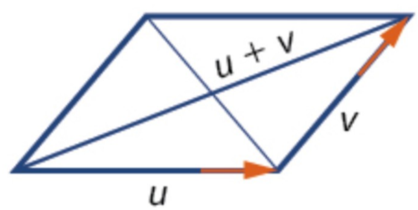


Figure 4: The sum of two vectors.

Hence, the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ form a triangle. How can we find the length of the side defined by $\mathbf{u} + \mathbf{v}$, i.e. the magnitude of $\mathbf{u} + \mathbf{v}$? One way to do it using dot products:

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = ||\mathbf{u}||^2 + 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2.$$

Another way is through the **Law of Cosines**: if $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2(||\mathbf{u}||)(||\mathbf{v}||) \cos \theta.$$

If we equate these two expressions, which are equal to each other, we get one of the most important properties of dot products:

**Theorem 1.** *For vectors* $\mathbf{u}$ *and* $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta$.

**Problem 7.** Show that, if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors, then $\mathbf{u} \cdot \mathbf{v} = 0$.

# 6 Matrices

**Definition.** A matrix is a rectangular array of numbers, called **entries**. A matrix has two dimensions: an $m \times n$ matrix has exactly $m$ rows and $n$ columns. We will notate matrices with $\mathbf{A}, \mathbf{B}, \ldots$. An example of a $3 \times 2$ matrix is shown below.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

One way of thinking about matrices is as an extension of vectors. Here, there can be as many columns as we want, while vectors are limited to a $m \times 1$ matrix. It follows that matrices exhibit many of the same operations as vectors, such as scalar multiplication, addition, and subtraction. There also exist some special properties of the matrix that is unique to itself, one of which we will introduce.

**Definition.** *(Determinant)* Let $\mathbf{A}$ be a matrix. The determinant of $\mathbf{A}$, denoted as $|\mathbf{A}|$, is a special value of a matrix that only exists for square matrices, or ones with $n \times n$ dimensions. We will define the determinant recursively given the value of $n$. In a $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have that $|\mathbf{A}| = ad - bc$.

In a $3 \times 3$ matrix $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|\mathbf{B}|$ is determined by iterating over the top row, taking each entry $x$ for example, and alternating adding and subtracting $x$ multiplied by the determinant of the $2 \times 2$ matrix formed by the entries not in $x$'s row or column. Thus,

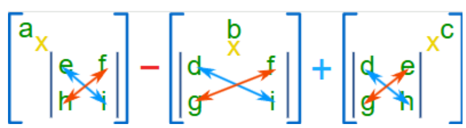

Figure 5: The determinant of a general $3 \times 3$ matrix.

$$|\mathbf{B}| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

It is known that $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$, or in other words, the determinant is multiplicative.

**Definition.** *(Matrix Multiplication)* Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{B}$ be a $n \times p$ matrix. Denote the entry in the $i$th row and $j$th column in matrix $\mathbf{A}$ $a_{ij}$ and in matrix $\mathbf{B}$ $b_{ij}$. $\mathbf{C} = \mathbf{AB}$ is then the $m \times p$ matrix

$$\mathbf{C} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \ldots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \ldots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \ldots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

Precisely,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

$c_{ij}$ can also be thought of as the dot product of the $i$th row in $\mathbf{A}$ and the $j$th column of $\mathbf{B}$.

A special case of matrix multiplication is when $m$ or $p$ is 1, indicating **vector-matrix multiplication** where the result is another vector.

**Definition.** *(Inverse Matrix)* The identity matrix $\mathbf{I}_n$ is a $n \times n$ matrix with all zeros except for its main diagonal being all ones.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The inverse of a $n \times n$ matrix $\mathbf{A}$ is $\mathbf{A}^{-1}$ is the unique matrix such that $\mathbf{A}\mathbf{A}^{\mathbf{-1}} = \mathbf{I}_n$.

**Problem 8.** Prove that $|\mathbf{A}| = \frac{1}{|\mathbf{A}^{-1}|}$.

**Problem 9.** Prove that for any real number $k$, $(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1}$.

**Problem 10.** Given $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find a general formula for $\mathbf{A}^{\mathbf{-1}}$. Prove exactly what matrices $\mathbf{A}$ do not have an inverse.

# 7 Parting Problems

**Problem 11.** For two matrices $\mathbf{A}$ and $\mathbf{B}$, is it true that $\mathbf{AB} = \mathbf{BA}$, as long as both products are defined? If so, explain why (no need for a formal proof); if not, give a counterexample.

**Problem 12.** Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Assuming that $\mathbf{A}^{-1}, \mathbf{B}^{-1}$, and $(\mathbf{AB})^{-1}$ all exist, find an expression for $(\mathbf{AB})^{-1}$ in terms of $\mathbf{A}, \mathbf{B}, \mathbf{A}^{-1}$, and $\mathbf{B}^{-1}$.

**Problem 13.** (Hint: think rotations, reflections, and translations.)

(a) Consider the matrix $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Geometrically, what does $\mathbf{M}$ map a vector in $\mathbb{R}^2$ to? In other words, what does the product $\mathbf{Mv}$ do to a vector $\mathbf{v}$? (No proof is necessary.)

(b) What about the matrix $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

**Problem 14.** We say that the **span** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is the set of all vectors that can be written as $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, where $a_1, a_2, \dots, a_n$ are scalars.

(a) When is the span of two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ equal to $\mathbb{R}^2$ (the entire plane), and when is it not?

(b) When is the span of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ equal to $\mathbb{R}^3$, and when is it not?

(c) Can you generalize to higher dimensions?

**Problem 15.** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ matrix. Find all vectors $\mathbf{v}$ in $\mathbb{R}^2$ for which $\mathbf{Av} = \mathbf{v}$.

# 8 Solutions

1. $B = (-2, -3)$.

2. We have points $O = (0, 0), A = (4, 3)$, and $B = (-2, -3)$. Thus, $OA = \sqrt{(4-0)^2 + (3-0)^2} = 5$ and $OB = \sqrt{(-2-0)^2 + (-3-0)^2} = \sqrt{13}$.

3. Let two vectors be $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ \vdots \\ b_n + a_n \end{bmatrix}$ which is true because addition itself is associative.

4. Let $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Of course, $a\mathbf{v} + b\mathbf{v} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} = \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix} = (a+b)\mathbf{v}$

5. Note that $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ are the sides of a triangle. This triangle has side lengths $||\mathbf{u}||, ||\mathbf{v}||$, and $||\mathbf{u} + \mathbf{v}||$, so of course the inequality $||\mathbf{u}|| + ||\mathbf{v}|| \geqslant ||\mathbf{u} + \mathbf{v}||$ holds by Triangle Inequality. The equality case is when the degenerate line is a triangle, so $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.

6. If $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2 = \left(\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 = ||\mathbf{v}||^2$.

7. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors, the angle between the two vertices is $90^\circ$. $\cos(90^\circ) = 0$, so $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(90^\circ) = 0$.

8. By definition, $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$, and it is given that determinants are multiplicative, so $|\mathbf{A}| \cdot |\mathbf{A}^{\mathbf{-1}}| = |\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{I}_n| = 1$ because the determinant of the identity matrix is 1. This gives $|\mathbf{A}| = \frac{1}{|\mathbf{A}^{\mathbf{-1}}|}$

9. Note that because multiplication is associative, $k^{-1}\mathbf{A}^{\mathbf{-1}}\mathbf{I_n} = k^{-1}\mathbf{A}^{-1}(k\mathbf{A})(k\mathbf{A})^{-1} = (k\mathbf{A})^{-1}$

10. $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The inverse does not exist if and only if $|A| = 0$.

11. No, it is not true. For example, if $\mathbf{A}$ is a $3 \times 4$ matrix and $\mathbf{B}$ is a $4 \times 5$ matrix, $\mathbf{AB}$ is a $3 \times 5$ matrix, but $\mathbf{BA}$ is a $5 \times 3$ matrix.

12. $\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{A}\mathbf{I_n}\mathbf{A}^{-1} = \mathbf{I_n}$, so the answer is $(\mathbf{B}^{-1}\mathbf{A}^{-1})$.

13. Rotation 180 about the origin (or reflect across both axes), and reflection over $y = x$.

14. For $\mathbb{R}^2$, it spans the plane when $\mathbf{u}, \mathbf{v}$ are not linearly independent, and only spans a line if they are. Same for $\mathbb{R}^{\nvDash}$ and the generalization.

15. Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We want to find all vectors $\mathbf{v} \in \mathbb{R}^2$ such that

$$\mathbf{A}\mathbf{v} = \mathbf{v}.$$

If we let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This can be rewritten as

$$\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Nontrivial solutions exist when the matrix

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}$$

is singular. This happens if

$$|(\mathbf{A} - \mathbf{I})| = (a-1)(d-1) - bc = 0.$$

Assuming that this determinant is zero (so that there is at least one nonzero solution), the solutions to

$$(a-1)x + by = 0$$

giving

$$y = -\frac{a-1}{b}x, \quad x \neq 0.$$

Thus, all such vectors are of the form:

$$\boxed{\begin{bmatrix} x \\ -\frac{a-1}{b}\,x \end{bmatrix}}$$

In other cases (such as $b = 0$), a similar parameterization can be derived. If the determinant condition is not met, the only solution is $\mathbf{v} = \mathbf{0}$.