MCAMC

Live Round

Middlesex County Academy Math Competition

April 12, 2025

This section of the competition is to be completed by **your team** within **1 hour**.

This section consists of **8 sets of 3 questions each**. You will receive each set only once you hand in the previous set.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

All answers are (possibly negative) **integers, fractions, and radicals**.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Best of luck!

1 Set 1

Team Name: ______________________

Team ID: ______________________

1. ________ 2. ________ 3. ________

1. There is a class with some number of boys and girls. If the number of boys is 10 less than the total number of students in the class, and the number of girls must equal half of the total number of students, how many total students are there?

2. When John works out, he exclusively lifts weights that are positive factors of 12. How many different weights can he lift?

3. At MCA, the vending machines use a very peculiar currency. They accept zigs, zags, and zigzags. If a zigzag is equal to 8 zigs and 3 zags, one zag is equal to 11 zigs, and one zig is equivalent to 2 USD, how much in American currency is 9 zigzags?

2 Set 2

Team Name: ______________________

Team ID: ______________________

4. ____________ 5. ____________ 6. ____________

4. There are 5 people making a speech on the MCA Career Day. The order in which the people speak will be randomly decided on the day of the event. Veda, Loki, and Venky are among the guest speakers. If the probability that Veda speaks before Loki, and Loki speaks before Venky in simplest form is $\frac{m}{n}$, what is the value of $m + n$?

5. When you fully expand $(a + b)^3$, what will the coefficient of the $a^2 \cdot b$ term be?

6. The English language can be seen as a hexavigesimal system (base 26). If the alphabet is assigned numerical values from A = 0 to Z = 25 in increasing sequence corresponding to their position in the alphabet, what is ACE + BAT translated from hexavigesimal to decimal form?

3 Set 3

Team Name: ______________________________

Team ID: ______________________________

7. ____________ 8. ____________ 9. ____________

7. In a horse racing competition with 30 total competitors. Each competitor team is able to use 2 different horses and 3 different jockeys, which are not shared with other teams. Find the total number of distinct horse-jockey combinations that he can compete against.

8. Ashwath wants to buy chicken nuggets from McAcademy. Yet, the eccentric fast food chain only sells nuggets in boxes of 5 and 4. What is the most number of nuggets that Ashwath cannot buy based on these restrictions?

9. Dip is waiting for English class to end. The class begins at 2:20 and ends at 3:10. If the angle between the hour and minute hand at 2:20 in degrees is a, and the angle between the hour and minute hand at 3:10 in degrees is b, what is $|a - b|$?

4 Set 4

Team Name: ______________________________

Team ID: ______________________________

10. __________ 11. __________ 12. __________

10. Muye flips a coin of diameter 1 onto an infinitely-sized chessboard with tiles of length 2. If the probability that the coin only touches a single tile is $\frac{a}{b}$ in simplest form, what is $a + b$?

11. A bookmaker accepts bets on the match between Team A and Team B. The bookmaker offers the following deal: If a bet is lost, he will provide 10% insurance on losses, but if a bet is won, he will award 90% of the total pool. He accepts \$700 in bets on Team A winning and he accepts \$400 in bets on Team B winning. If Team A and Team B each have a 50% chance of winning, what are the expected earnings of the bookmaker?

12. Find the smallest positive integer n such that $9n-2$ and $7n+3$ share a common factor greater than 1.

5 Set 5

Team Name: ______________________________

Team ID: ______________________________

13. ____________ 14. ____________ 15. ____________

13. Nate and Eli are rivals in an auction for the coveted MCA hoodie. Nate and Eli both start bidding at \$0 and can keep bidding until they reach the maximum price of \$363. Nate and Eli can both increase the bid by \$1, \$2, \$3, \$4, \$5, or \$6. Nate goes first. What should Nate bid on the first try to guarantee a win?

14. Find the number of distinct fractions between 0 and 1 that can be written with denominator 2025 in reduced form.

15. Three concentric circles are drawn such that each successively larger circle has a circumference equal to the area of the smaller circle. Given that the largest circle has a radius of 98, what is the ratio of the area of the outer ring to the area of the smallest circle?

6 Set 6

Team Name: ____________________

Team ID: ____________________

16. ________ 17. ________ 18. ________

16. Compute the remainder when 2^{30} is divided by 31.

17. Alexander conducted a survey on 100 people. 56 of them have binders, 63 of them have folders, 87 of them have notebooks, and 98 of them have pencils. What is the minimum number of people that have all 4 items?

18. Sahya is a snake that is 8 units long. His head and tail are both on the bottom-left corner of a square grid with 6 dots in each row and column (a 6×6 grid), and he can move his head up by 1 or to the right by 1 until his entire body is within the grid (while keeping his tail at the origin). In how many distinct ways can Sahya's body lay on the grid?

7 Set 7

Team Name: ______________________

Team ID: ______________________

19. ________ 20. ________ 21. ________

19. A square lying on the coordinate plane entirely in the first quadrant with sides parallel to the axes has one vertex at the origin and another vertex on the parabola $y = x^2 - 12x + 36$. Find the sum of the areas of all possible such squares.

20. Three circles are drawn such that they are each externally tangent to the other two circles and their radii form an arithmetic sequence. The area of the right triangle formed by their centers is 486. Find the product of the greatest common denominator (GCD) and the least common multiple (LCM) of their radii.

21. Pac-Man starts at the center of an array of infinite dots spaced 1 unit apart. Due to a machine malfunction, Pac-Man can only move 1 unit up or 1 unit to the right each time. You move Pac-Man 41 times before the Ghost appears and begins chasing Pac-Man down in a straight line from the center. What is the shortest possible distance that the Ghost must travel to catch Pac-Man?

8 Set 8

Team Name: ________________________________

Team ID: ________________________________

22. ____________ 23. ____________ 24. ____________

22. In triangle ABC, the medians from vertices A, B, and C have lengths 5, 6, and 7, respectively. Find the area of triangle ABC.

23. There are 3 distinct positive integers A, B, and C with $A < B < C$, each less than 50, such that each one of the 3 pairwise possible sums are consecutive perfect squares. Find the maximum value of $A + B + C$.

24. Suppose that there are 5 colors of T-shirts and you can draw as many T-shirts of each color as necessary. You draw T-shirts one by one at random. What is the expected number of T-shirts that you need to draw until you have a complete set (at least one of each color)?

9 Solutions

1. If half the class is girls, the other half must be boys. This means that half the class is 10 less than all of the class, so the answer is $10 \cdot 2 = \boxed{20}$.

2. All factors of 12 are less than or equal to 12, so we only need to check 12 numbers. These are precisely $1, 2, 3, 4, 6, 12$, so the answer is $\boxed{6}$

3. If one zig is 2 USD, one zag is $11 \cdot 2 = 22$ USD, and one zigzag is $8 \cdot 2 + 3 \cdot 22 = 82$ USD. Therefore, 9 zigzags is $9 \cdot 82 = \boxed{738}$ USD.

4. The only thing that matters is the order Veda, Loki, and Venky speak, their order relative to the other people do not matter. Therefore, given the 6 possible permutations of the order of Veda, Loki, and Venky, only one fits the conditions. Thus, the probability is $\frac{1}{6}$ and the answer is $1 + 6 = \boxed{7}$.

5. For the nth term in the expanded form of $(a + b)^m$, the coefficient of the term $a^n b^{m-n}$ will be $\binom{m}{n}$. Therefore, the coefficient of the sixth term will be $\binom{3}{2} = \boxed{3}$.

6. Recall that the rightmost place value is the ones place, or 26^0, the second place from the right is 26^1, and the third 26^2 when converting to decimal values. Assigning letters their corresponding values is fairly simple, but keep in mind that the letters are assigned one less than their position in the alphabet, e.g. E = 4, not 5. Using the place value conversions, we obtain $(0 + 1) \cdot 26^2 + (2 + 0) \cdot 26 + (4 + 19) \cdot 1 = \boxed{751}$.

7. Each competitor team is able to create $2 \cdot 3$ different horse-jockey combinations, so the answer is $(2 \cdot 3) \cdot 30 = \boxed{180}$

8. If x represents horizontal units and y represents vertical units of Pac-Man's position, the distance that the Ghost travels is $\sqrt{x^2 + y^2}$. To minimize this distance, the Ghost needs to travel closest to the diagonal $y = x$, however since $x + y = 41$ and both x and y have to be positive integers, the closest two numbers that satisfy this are 20 and 21. Thus the distance is $\sqrt{20^2 + 21^2} = \sqrt{841} = \boxed{29}$ units.

9. Angles (in degrees) between clock hands can be calculated by multiplying the value of the hour by 30, multiplying the value of the minute by $\frac{11}{2}$, and finding the absolute value of the difference of the two values. Thus, the angle at 2:20 will be $|2 \times 30 - 20 \times \frac{11}{2}| = 50$, and the angle at 3:10 will be $|3 \times 30 - 10 \times \frac{11}{2}| = 35$. Thus, the answer is $|50 - 35| = \boxed{15}$

10. In order for the coin to only touch a single tile, it cannot overlap with other tiles. This means the coin can either land directly on the center of a tile, land such that its circumference touches the perimeter of a given tile, or anywhere in between. Reducing the coin to a single point A located at its center, we can determine the area in which point A is bound to, which is a square of length 1 with its center point at the center point of any given tile. If point A falls within this square, the coin meets the criteria, otherwise it doesn't, and the probability of point A falling within this square is what we are looking for. As this area is exactly $\frac{1}{2}^2 = \frac{1}{4}$ the area of any given tile, and since each tile has this smaller square within it, the probability of a coin meeting the aforementioned criteria is $\frac{1}{4}$, and so the solution is $\boxed{5}$.

11. Although the differences are both 10%, the deduction of the payout includes a deduction on the original bet. Out of the total pool of \$1100, the bookmaker returns 90% of the total pool to the winning bettors and keeps 10% of the total pool, which is equal to 10% of the losing bet + 10% of the winning bet. Then, he insures 10% of the losing bet to the losing bettors, leaving 10% of the winning bet unreturned. It turns out that 10% of the winning bet value is pocketed by the bookmaker. Since there is an equal chance of Team A or Team B winning, the expected value is thus $0.5 \cdot \$70 + 0.5 \cdot \$40 = \$55$, $\boxed{55}$.

12. Using the Euclidean Algorithm, we see that the two final terms are -41 and $n + 18$. Since 41 is prime, $n + 18$ needs to be divisible by 41, and the smallest positive integer that satisfies the expression is $\boxed{23}$.

13. If the maximum bid is a multiple of 7, Eli can counter Nate, because if Nate bids n, Eli bids $7 - n$, always landing on a multiple of 7. If the maximum is not a multiple of 7, Nate can play some amount to be a multiple of 7 away from the maximum bid, flipping him and Eli's roles and giving Nate the win. Since 363 mod 7 is 6, Nate, to win, needs to play a starting bid of $\boxed{6}$.

14. This is equivalent to finding the number of integers less than 2025 which are relatively prime to 2025, which we can do using Euler Totient Function. The prime factors of 2025 are 3 and 5. Therefore, $\phi(2025) = (1 - \frac{1}{3})(1 - \frac{1}{5})(2025) = \boxed{1080}$.

15. The area of the largest circle is $98^2\pi$ and its circumference is $98 \cdot 2\pi = 192\pi$ which is the area of the medium circle, and thus the circumference of the medium circle is $\sqrt{196} \cdot 2\pi = 28\pi$, which is the area of the smallest circle. Since the desired value is the difference of areas between the largest and medium circles divided by the area of the smallest circle, we can obtain $\frac{(98 \cdot 98 - 98 \cdot 2)\pi}{28\pi} = \frac{98 \cdot 96}{28} = \frac{14 \cdot 7 \cdot 24 \cdot 4}{7 \cdot 4} = \boxed{336}$.

16. Fermat's Little Theorem states that $a^{p-1} \bmod p \equiv 1$ for any prime number p and a relatively prime. As 31 is a prime number, and $2 \bmod 31 \neq 0$, our problem conveniently works out to equal $\boxed{1}$.

17. To minimize the number of people that have every item, we need to minimize the number of people that have no items and maximize the number of people that have only three items. We can do this by overlapping the people that don't have a particular item, with the people that have another item. Doing this for all 4 items, we will get $56 - (100 - 63) - (100 - 87) - (100 - 98) = \boxed{4}$.

18. Since Sahya is 8 units long, he will make 8 moves total, which can either be up (U) or right (R). Since the grid is 6×6 and Sahya starts at the coordinates $(1, 1)$, he cannot make more than 5 moves up or right. Therefore, Sahya's 8 moves can either be 5U and 3R, 4U and 4R, or 3U and 5R. Since up and right moves are identical to other up and right moves respectively, we can use the choose function to determine the number of unique combinations of these moves. The total number of combinations will be $\binom{8}{3} + \binom{8}{4} + \binom{8}{5}$, with the bottom number representing the number of R moves Sahya makes. Adding these values together, we get a sum of $\boxed{182}$.

19. If the square lies entirely in the first quadrant, there must be one other vertex on the x-axis and one other vertex on the y-axis. Therefore, the vertices of the square, given that it has side

length s, are $(0,0),(0,s),(s,0),and(s,s)$. Case 1: $(0,s)$ on the parabola - Then $s = 36$ and the square has area $36^2 = 1296$. Case 2: $(s,0)$ on the parabola - Then $s^2 - 12s + 36 = 0$ and $s = 6$, so the square has area $6^2 = 36$. Case 3: (s,s) on the parabola - Then $s = s^2 - 12s + 36$, $s^2 - 13s + 36 = 0$, and $s = 4, 9$. The two possible areas of the squares are 16 and 81.

The sum of all areas is $1296 + 36 + 16 + 81 = \boxed{1429}$.

20. The side lengths of the right triangle are the sums of two radii. Setting the smallest radius to n, the two larger radii are $n + r$ and $n + 2r$. The two smaller side lengths must be $2n + r$ and $2n + 2r$. Since the hypotenuse is $2n + 3r$, Pythagorean theorem can be applied: $(2n+r)^2 + (2n+2r)^2 = (2n+3r)^2 = 8n^2 + 12nr + 5r^2 = 4n^2 + 12nr + 9r^2$. Simplifying this, we obtain $4n^2 = 4r^2$, and thus $n = r$. This means that the two legs of the right triangle are $3n$ and $4n$, and $6n^2 = 486$, so $n = 9$. Once the radii are determined to be 9, 18, and 27, the GCD of this set is 9, and the LCM of this set is 54. Coincidentally, their product is $\boxed{486}$.

21. This problem is an example of the McNugget Theorem, or the Frobenius Coin Problem. This theorem states that for any two relatively prime positive integers a and b, the greatest integer that cannot be written as the sum of non-negative multiples of a or b is $ab - a - b$. Plugging in 4 and 5, the solution will be $\boxed{11}$.

22. The three medians of a triangle form a triangle with side lengths 5, 6, and 7.We can find the area of the median triangle. Using Heron's formula, first calculate the semiperimeter s:

$$s = \frac{5+6+7}{2} = 9.$$

Then, the area of the median triangle, K_m, is:

$$K_m = \sqrt{s(s-5)(s-6)(s-7)} = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = \sqrt{216} = 6\sqrt{6}.$$

The area of the original triangle is $\frac{4}{3}$ times the area of the median triangle. Therefore, the area K of triangle ABC is:

$$K = \frac{4}{3}K_m = \frac{4}{3}(6\sqrt{6}) = 8\sqrt{6}.$$

So, the area of triangle ABC is $\boxed{8\sqrt{6}}$.

23. Assume there exist positive integers s, t, and u such that

$$x + y = s^2, \quad x + z = t^2, \quad y + z = u^2.$$

Adding these equations gives

$$2(x + y + z) = s^2 + t^2 + u^2,$$

so that

$$x + y + z = \frac{s^2 + t^2 + u^2}{2}.$$

Meaning we need to find the sum

$$S = \frac{s^2 + t^2 + u^2}{2}.$$

We can express each variable in terms of s, t, and u:

$$x = \frac{s^2 + t^2 - u^2}{2}, \quad y = \frac{s^2 + u^2 - t^2}{2}, \quad z = \frac{t^2 + u^2 - s^2}{2}.$$

Since x, y, and z must be positive, distinct, and less than 50, we must choose s, t, and u so that they fit those conditions. Since the numbers have to be consecutive a put of testing leads to $s = 7$, $t = 8$, $u = 9$ being the maximum value. Then

$$7^2 + 8^2 + 9^2 = 49 + 64 + 81 = 194,$$

and

$$S = \frac{194}{2} = \boxed{97}$$

24. The total number of T-shirts needed is just the number of T-shirts needed to see the first new color plus the number of T-shirts needed to see the second new color after the first color, and so on. If N_j represents the expected additional amount you must draw after the first instance of the $(j-1)$-th color up until and including the first instance of the j-th color, then by linearity of expectation the total expected number of T-shirts you must draw is $\sum_{j=1}^{5} N_j$. In the range of draws after the first instance of the $(j-1)$-th color up until and including the first instance of the j-th color, the probability of drawing a new color T-shirt on each draw is $\frac{5-(j-1)}{5}$ since $j-1$ colors have already been drawn. Let $p = \frac{5-(j-1)}{5}$. Then $N_j = p \cdot 1 + (1-p) \cdot (1 + N_j)$ by considering both cases. Rearranging allows us to rewrite as:

$$\begin{aligned} N_j &= p \cdot 1 + (1-p) \cdot (1 + N_j), \\ N_j &= p \cdot 1 + (1 + N_j) - p - pN_j, \\ pN_j &= 1, \\ N_j &= \frac{1}{p}, \\ N_j &= \frac{5}{5-(j-1)} \end{aligned}$$

Then the final probability becomes

$$\begin{aligned} \sum_{j=1}^{5} N_j &= \sum_{j=1}^{5} \frac{5}{5-(j-1)} \\ &= \sum_{k=0}^{4} \frac{5}{5-k} \\ &= 5\left(1 + \frac{1}{2} + \cdots + \frac{1}{5}\right) \\ &= 5\left(\frac{60}{60} + \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60}\right) \\ &= \boxed{\frac{137}{12}} \end{aligned}$$