Individual Round

Middlesex County Academy Math Competition

April 12, 2025

This section of the competition consists of **25 questions** to be completed **individually** within **1 hour**.

Questions are **not necessarily** in order of increasing difficulty.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

All answers are (possibly negative) **integers, fractions, and radicals**.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Best of luck!

1 Answer Sheet

Name: ____________________

Individual ID: ____________________

Team Name: ____________________

Team ID: ____________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________
7. ________
8. ________
9. ________
10. ________
11. ________
12. ________
13. ________
14. ________
15. ________
16. ________
17. ________
18. ________
19. ________
20. ________
21. ________
22. ________
23. ________
24. ________
25. ________

2 Problems

1. Calculate $1 + 2 + 3 + \cdots + 25$.
2. In MCA-land, each minute has 3 seconds, each hour has 4 minutes, and each day has 5 hours. How many seconds are in 6 days?
3. When Jane was born, her mother planted a very special apple tree. It only produces apples once every year, on Jane's birthday, and the number of apples that fall is equal to the age Jane is turning. For example, on Jane's 17th birthday, 17 apples will fall. How many total apples will have fallen by the time Jane has turned 75?
4. Given 5 positive integers such that the unique mode is 4, the median is 3, and the average is less than 3, find the sum of all 5 integers.
5. If positive integers x, y satisfy $x + y = 17$ and $xy = 52$, find $x^2 + y^2$.
6. A hexagon is split into 6 congruent equilateral triangles, each with a side length of 2. What is the square of the hexagon's area?
7. Find the smallest positive integer that has a remainder of 5 when divided by 7, a remainder of 4 when divided by 6, and a remainder of 8 when divided by 10.
8. Each day, Aarush wakes up and flips a coin that has a $\frac{2}{3}$ chance to land on heads and $\frac{1}{3}$ chance to land on tails. What is the probability that Aarush flips 2 heads and 2 tails in 4 days?
9. Aarav drives 40 mph to get to work in the morning when there is a lot of traffic. He leaves work early and drives home at an average speed of 60 mph. What was his average speed for the whole trip?
10. The special operation $\rightsquigarrow$ satisfies $x \rightsquigarrow x = 1$ and for $x > y$, $x \rightsquigarrow y = x \cdot y + y \rightsquigarrow (x - y)$. Find $21 \rightsquigarrow 13$.
11. A 4-digit palindrome is a number that reads the same forwards and backwards (for example, 1221 or 3443). Such a number can be written in the form ABBA, where A and B are digits and A $\neq$ 0. Determine how many 4-digit palindromes are divisible by 9.
12. The positive integer side lengths of a right triangle are a, b, c, where c is the hypotenuse. Find the smallest value of c such that there exist exactly 4 possible values for a.
13. Alice and Bob are playing a series of rock paper scissors, first to 2 wins. Assuming they draw exactly once, and Alice eventually wins, how many different series were possible? (Two series are distinct if at any point, a player makes a different move).
14. A parallelogram has side lengths 4, 6, 4, 6. What is the maximum area of the parallelogram?
15. Define the sequence a_n by $a_1 = 3, a_2 = 7$ and for $n \geqslant 3, a_n = a_{n-1} - a_{n-2}$. Find a_{100}.
16. MCA is restocking the vending machine. They can fill up to 19 bags of Doritos and up to 14 bags of Cheetos. However, they cannot fit more than 30 items in the vending machine. The vending machine is not necessarily full after restocking. How many different ordered pairs of Doritos and Cheetos can they put in the vending machine ?

17. Given $\text{lcm}(x, y) = 50$ and $\gcd(x, y) = 10$, find the sum of all possible values of $x + y$.

18. Find the number of integers of the form $a^b < 500$ where a and b are positive integers with $b \geqslant 2$.

19. Consider a box that is 12 inches wide, 16 inches long, and 20 inches high. What is the area, in square inches, of the largest rectangular sheet of cardboard that can fit inside this box? Assume the cardboard cannot bend and is of negligible thickness.

20. For a positive integer n, let $f(n)$ be the number of 1's in the binary (base-2) representation of n. For example, since

$$14_{10} = 1110_2,$$

we have $f(14) = 3$. Determine the number of integers n with $1 \leqslant n \leqslant 100$ for which $f(n) = 3$.

21. Sam is throwing darts on a dartboard. Sam can throw the dart on tiles from 1 to 20 points. If the probability the dart hits each tile is equal, and the probability that he scores exactly 49 points after 3 shots is $\frac{a}{b}$, what is the value of a?

22. Evaluate

$$\log_3(304) \cdot \frac{\log_{304}(27) + \log_3(81)}{\log_{27}(304) + \log_{81}(3)}.$$

23. Let the *1-box* be a cube of side length 1, and the *1-sphere* be the sphere inscribed in the 1-box. For $n > 1$, the *n-box* is the cube inscribed in the $(n-1)$-sphere, and the *n-sphere* is the sphere inscribed in the $(n-1)$-box. Find the ratio between the radius of the 1-sphere and the 10-sphere.

24. If $a_{n+1} = a_n - 9$ and $a_0 = 405$, find the value of

$$-\sum_{k=0}^{45} \frac{9}{(a_k)^2 + 5a_k - 14}.$$

25. There are 9 seats at a seminar at Edison Academy, arranged in a 3 x 3 array. There are 3 juniors, 2 seniors, and 1 freshman. How many seating arrangements are possible if

   (a) If all students must have a seat,

   (b) If no students of the same grade may sit in the same row,

   (c) If the freshman is too scared to sit directly to the right or left of a senior?

3 Solutions

1. We will use the formula for an arithmetic sequence, which gives $\frac{25\cdot 26}{2} = \boxed{325}$

2. To find out how many seconds there are in some amount of days, we can first figure out how many seconds there are in a minute, then an hour, then a day, and then multiply it by the number of days. There are 3 seconds in a minute, $3 \cdot 4 = 12$ seconds in an hour, $12 \cdot 5 = 60$ seconds in a day, and finally $60 \cdot 6 = \boxed{360}$ seconds in 6 days.

3. The number of produced apples starts at 1 and grows by 1 every year until 75. Effectively, the number we are looking for is the sum of the first 75 positive integers. The sum of the first n positive integers $\frac{n(n+1)}{2}$, so the the answer we are looking for is $\frac{75\cdot 76}{2}$, which becomes $\boxed{2850}$.

4. The sequence is of odd length so the middle term must be 3. Then, the two biggest terms must be 4, and the two smallest terms must be distinct because 4 is the unique mode. This gives that the two smallest terms must be 1 and 2, because the sequence is of positive integers. The sequence must be 1, 2, 3, 4, 4 by definition, so the answer is $\boxed{14}$.

5. Using a common algebraic substitution, we find that $x^2 + y^2 = (x + y)^2 - 2xy = 17^2 - 52 = 289 - 2 \cdot 52 = \boxed{185}$. Alternatively, one can figure out that $(x, y) = (13, 4)$ and solve from there.

6. Setting each equilateral triangle's area to A, the hexagon's area is $6A$, and the desired value is thus $36A^2$. To find the height of the equilateral triangle of side length 2, we draw a bisecting altitude and obtain two congruent $30 - 60 - 90$ triangles. Using Pythagorean theorem, we see that the height is $\sqrt{2^2 - 1^2} = \sqrt{3}$. Then using the triangle area formula, $A = \frac{2\sqrt{3}}{2} = \sqrt{3}$. Plugging A into $36A^2$, the answer is $36 \cdot 3 = \boxed{108}$.

7. If a number x satisfies the constraints, then $x + 2$ is divisible by $\text{lcm}(6, 7, 10) = 210$, so we are looking for $\text{lcm}(6, 7, 10) - 2 = 210 - 2 = \boxed{208}$.

8. There are $\binom{4}{2} = 6$ ways to decide on which days Aarush flips heads and which days Aarush flips tails. Then, no matter the order, he has a $\frac{2}{3}$ chance of flipping heads on the days we decided, and a $\frac{1}{3}$ chance of flipping tails on the days we decided, so we are looking for

$$\binom{4}{2} \cdot (\frac{2}{3})^2 \cdot (\frac{1}{3})^2 = \frac{6 \cdot 4 \cdot 1}{3^4} = \boxed{\frac{8}{27}}$$

9. Take the harmonic mean of 40 and 60 to get the answer of $\boxed{48}$. Alternatively, assume it takes 1 hour to get to work, so work is 40 miles away. Then it will take $\frac{40}{60} = \frac{2}{3}$ the amount of time to get back home. Therefore, we've traveled 80 miles in $\frac{5}{3}$ hours, giving 48 as our answer as well.

10. Simply list out the steps until we get to the case $1 \rightsquigarrow 1 = 1$, where we can extract the answer $21 \cdot 13 + 13 \cdot 8 + 8 \cdot 5 + 5 \cdot 3 + 3 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 = \boxed{441}$

11. A 4 digit palindrome $ABBA$ has digit sum $A+B+B+A = 2(A+B)$. If a number is divisible by 9, then the sum of its digits has to be divisible by 9. Since 2 and 9 have no common factors, this means that $A + B$ must be divisible by 9. Since $1 \leqslant A \leqslant 9$ and $0 \leqslant B \leqslant 9$./ There are 2 cases, either $A + B = 9$ or $A + B = 18$. For the first case, for each value A from 1 to 9 there is 1 and only 1 respective B value that works, meaning there are 9 possibilities. For the second case, only $A = 9$ and $B = 9$ work. So in total there are $9 + 1 = \boxed{10}$ possibilities.

12. Note that if (a, b) is a solution, so is (b, a), so we really only need c to be the hypotenuse of two non-congruent right triangles. The smallest Pythagorean primitive triples are 3-4-5, 5-12-13, and 7-24-25, so we can see that $\boxed{25}$ works and no smaller integer works (15-20-25 and 7-24-25).

13. There are 3 ways to win, 3 ways to lose, and 3 ways to draw rock paper scissors. If Alice wins 2-0, then there are two wins and one draw in the series. There are $\binom{3}{1} - 1 = 2$ ways to choose which game is a draw (it can't be the last game, because the series would have ended already), so this case is $2 \cdot 3^3 = 54$. If Alice wins 2-1, there are 3 ways to choose the draw and 2 ways to choose which game is a loss, so this case is $3 \cdot 2 \cdot 3^4 = 486$, so $54 + 486 = \boxed{540}$.

14. Knowing that a quadrilateral has area maximized when cyclic, or considering that the height shrinks (by Pythagorean Theorem) as the parallelogram becomes more slanted, the area is maximized as a rectangle, which is just $4 \cdot 6 = \boxed{24}$

15. We can start solving the first few terms to look for a pattern:

$$\begin{aligned}
a_1 &= 3 \\
a_2 &= 7 \\
a_3 &= a_2 - a_1 = 7 - 3 = 4 \\
a_4 &= a_3 - a_2 = 4 - 7 = -3 \\
a_5 &= a_4 - a_3 = -3 - 4 = -7 \\
a_6 &= a_5 - a_4 = -7 - (-3) = -4 \\
a_7 &= a_6 - a_5 = -4 - (-7) = 3 \\
a_8 &= a_7 - a_6 = 3 - (-4) = 7
\end{aligned}$$

We find that $a_7 = 3 = a_1$ and $a_8 = 7 = a_2$. Meaning the sequence repeats with a period of 6. Since the sequence is periodic, we find the matching term for a_{100} is 100 mod 6 = 4, which means $a_{100} = a_4$. Since, $a_4 = -3$.

$$a_{100} = \boxed{-3}$$

16. Notice how if they fill 0 Doritos in the vending machine, they can fill anywhere from 0 to 14 Cheetos bags. This gives them 15 possible ordered pairs in this case. This pattern continues when they fill 1 Doritos bag, 2 Doritos bags, and so on up to 16 Doritos bags - for each of these values, they can still fit 0 to 14 Cheetos bags (15 ordered pairs each).When they reach 17 Doritos bags, they can only fit up to 13 Cheetos bags before hitting the 30-item limit, which gives 14 ordered pairs.With 18 Doritos bags, they can only include up to 12 Cheetos bags, giving 13 ordered pairs.Finally, with 19 Doritos bags, they can only include up to 11 Cheetos bags, giving 12 ordered pairs.Therefore, the total number of possible ordered pairs is $15 \cdot 17 + 14 + 13 + 12 = 255 + 39 = \boxed{294}$ different combinations of Doritos and Cheetos bags.

17. $\text{lcm}(x, y) \cdot \gcd(x, y) = xy$. Then we have that$xy = 500$ and $x, y >= \gcd(x, y)$ so we just check pairs of factors where both numbers are at least 10. This gives us the four pairs (10,50), (20,25), (25,20), and (50,10). The only pairs that satisfy the LCM and GCD properties are (10,50) and (50,10), making the only possible sum $\boxed{60}$

18. There are 22 perfect squares less than 500 ($22^2 = 484, 23^2 = 529$). There are 7 perfect cubes ($7^3 = 343, 8^3 = 512$), but we exclude 1^3 and 4^3 since those are also perfect squares. All of the powers of four are also powers of two so we skip those. There are 3 powers of five ($3^5 = 243, 4^5 = 1024$) but we exclude 1^5 since it is already counted. Powers of 6 and 8 were already covered by the squares, and the highest power we can consider is 8 since $2^9 = 512$. So we have to check powers of 7, which gives only one unique possibility of 2^7. Therefore the answer is $22 + 7 - 2 + 3 - 1 + 1 = \boxed{30}$.

19. With a little intuition, we can determine that the area is maximized when the cardboard rectangle along some diagonal. This is trivially larger than placing it on any flat face of the box, and gives us the formula $c\sqrt{a^2 + b^2}$ where a, b, c are the sides. Bringing c into the square root and distributing allows us to check that this is maximized when $c > a, b$. In other words, one side length of the rectangle is the largest side of the box and it is placed along the diagonal of the other two. Plugging in, we get $20\sqrt{12^2 + 16^2} = \boxed{400}$

20. Any positive integer n between 1 and 100 can be a 7 bit binary number because $100 \leqslant 2^7 - 1$. For a number with exactly 3 ones, we can choose 3 out of the 7 spots to be one meaning $\binom{7}{3} = 35$. However, this counts all binary numbers from 0 to 127. We need those from $1 \leqslant n \leqslant 100$. So we must account for overcounting. Only $64 + 32 + 16$ and $64 + 32 + 8$ result in values over 100, meaning we have to remove these cases, leading to $35 - 2 = \boxed{33}$ numbers.

21. The number of ways to add three numbers, each from 1 to 20, and receive a sum of 3, the minimum score, and 60, the maximum score, are the same, with there being exactly 1 way to do so. The same is true for the second smallest score, 4, and the second largest score, 59, both having 3 ways, showing the symmetry of these sums. We can also see that the number of ways to score 3 total points after three shots is equal to the first triangular number, 4 points has the second triangular number of ways, and so forth. Since 49 is the $12th$ number from the right, the number of ways a score of 49 can be achieved is the $12th$ triangular number, which can be calculated using the formula $\frac{n(n+1)}{2}$. Substituting in 12 for n, the number of ways to score exactly 49 points will be $\boxed{78}$.

22. By the change of base rule, $\log_a b \cdot \log_b a = 1$, and thus $\log_a b = 1/\log_a b$. Using this, we rewrite the problem as

$$\log_3(304) \cdot \frac{\log_{304}(27) + \log_3(81)}{\frac{1}{\log_{27}(304)} + \frac{1}{\log_{81}(3)}} = \log_3(304) \cdot \frac{\log_{304}(27) + \log_3(81)}{\frac{\log_{304}(27) + \log_3(81)}{\log_{304}(27) \cdot \log_3(81)}} = \log_3(304) \cdot \log_{304}(27) \cdot \log_3(81)$$

From here, change of base rule is applied to obtain $\log_3(27) \cdot \log_3(81) = 3 \cdot 4 = \boxed{12}$.

23. Let the ratio of the radius between the 1-sphere and the 2-sphere be r. Then the ratio between the 2-sphere and the 3-sphere is the same, and so on, so we are looking for r^{10}.

Assume the 1-sphere has radius x. Then the long diagonal of the 2-box inscribed inside the 1-sphere is equal to the diameter of the 1-sphere, which is $2x$. If the side length of the 2-box is s, then the long diagonal is $\sqrt{s^2 + s^2 + s^2} = s\sqrt{3}$, so $s\sqrt{3} = 2x$ and $s = \frac{2}{\sqrt{3}}x$.

Finally, the 2-sphere inscribed inside inside the 2-box has diameter equal to the side length of the 2-box, which is just s, so the radius of the 2-sphere is $\frac{s}{2} = \frac{x}{\sqrt{3}}$. Thus, $r = \sqrt{3}$.

The ratio of the radii decrease by $\sqrt{3}$ every time, so $\sqrt{3}^{10} = \boxed{243}$

24. By partial fractions and telescoping the answer becomes $\frac{1}{(a_0+7)} - \frac{1}{a_{45}-2} = \frac{1}{412} - \frac{1}{405-2-45\cdot 9} = \frac{1}{412} - \frac{1}{-2} = \frac{1}{412} + \frac{1}{2} = \boxed{\frac{207}{412}}$.

25. Determine cases for the juniors and freshman, and then calculate the possible number of arrangements for the seniors. There must be one junior per row. Then, when placing the freshman, either the freshman can be in a row where the junior occupies the middle seat, or a row where the junior occupies an edge seat.

    In the first scenario, there are 3 ways to choose the junior to sit next to the freshman, 2 ways to determine if the freshman will sit to the left or the right of the junior, 3 ways to determine the row in which the freshman will be sitting, 6 ways to place the second junior, and 3 ways to place the final junior, there are a total of $3 * 2 * 3 * 6 * 3 = 324$ ways to place the juniors and the freshman. The seniors can now be placed in any of the remaining seats, as the freshman occupies an edge seat and the junior the corresponding middle seat leaving no threat of the senior sitting next to the freshman. There are 5 remaining seats and 2 seniors, so there are $4 * 2 + 2 * 4$ ways to place the seniors, with either none of the seniors sharing a row with the freshman or a senior sharing the row with the freshman. This leaves $324 * 16 = 5184$ total seats for the preliminary case.

    As for the second scenario, there are 2 subcases; either the freshman will be on the edge or in the middle.

    For the first subcase, there are 6 seats for the freshman. There are 3 choices for the junior to sit in the opposite edge seat, 6 ways to place the second junior, and 3 ways to place the final junior. There is therefore a total of $6 * 3 * 6 * 3 = 324$ possibilities. Now, the seniors may not share a row with the freshman, as they would be placed in the middle seat next to the freshman. They must sit in the vacant seats of the other two rows, leaving $4 * 2 = 8$ possibilities. This leaves a total of $324 * 8 = 2592$ possibilities.

    For the second subcase, there are 3 seats for the freshman, 3 choices for the junior with whom they will share a row, 2 choices for the side of the freshman on which the junior will sit, 6 choices for the second junior, and 3 for the last, once again leading to 324 possibilities. With the seniors once again not being able to share a row with the freshman, there are still 8 possibilities for their seats, leading to another 2592 possibilities.

    This leads to a total of $5184 + 2592 + 2592 = \boxed{10368}$ solutions as all cases are mutually exclusive.