Team Round

Middlesex County Academy Math Competition

April 13, 2024

This section of the competition consists of **15 questions** to be completed by **your team** within **1 hour**.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

Answers must be **exact** (do not approximate π) and in **simplest form**, with all fractions expressed as improper fractions.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Some questions will require a brief explanation. Additionally, questions may have no answer. If so, the correct response is **"None"**. Problems that are not proof-based do not need an explanation.

A note on proofs: The proofs in this team round are designed so that **you will not need more than** 5 **or** 6 **sentences**. A good explanation should be brief, to the point, and easy to follow.

Best of luck!

1 Answer Sheet

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1.

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2 Prologue

This round focuses on a subject known as **Axiomatic Number Theory**, the basis of Number Theory.

Number Theory (NT) is one of the oldest fields in mathematics. You've undoubtedly used NT when doing any form of math. For example, suppose you're solving the equation

$$2x + 1 = 5$$

for x. Well, usually you'd subtract one from both sides to get $2x = 4$, then divide both sides by 2 to get $x = 2$.

But what is this operation known to everyone as subtraction? One might say "subtracting is the same as taking away something." But how do we subtract a negative number, then? How can we take away something negative from a number?

Division is even harder to explain. How do you define dividing 5 by 3? And how do you explain something like

$$\frac{5}{3} = 1.66\ldots?$$

How can a number have infinitely many digits? It can get very confusing very quickly, without some sort of order.

Number Theory exists to provide this structure to an otherwise free world of numbers. It's the law of mathematics, the rules that everybody has to follow if they want to use it. Without Number Theory, using numbers would be meaningless.

The point of this round is not to give you an existential crisis about arithmetic. In fact, it's quite the opposite. This round will give you a perspective into the building blocks of mathematics that many people take for granted. You'll see what the first mathematicians who came up with the number system we use everywhere today saw. And it all orignates from 6 facts.

3 The Foundation

Note 1: You are allowed to assume the regular properties of the equals sign, $=$. If you're interested, $=$ is known as an equivalence relation, which has three properties:

1. $a = a$
2. $a = b$ *implies* $b = a$
3. $a = b$ *and* $b = c$ *imply* $a = c$

Note 2: For this section ("The Foundation"), cite the axioms when appropriate and necessary. For example, if you are going from a step to another with something like $a \cdot (b+c) = a \cdot b + a \cdot c$, mention the word "distributivity," or cite "Property 5" or "Axiom 5".

You are likely familiar with the set of **integers**, denoted by $\mathbb{Z}$, even if you don't know it by name. Informally, integers are the infinite sequence

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

One of the large goals of Number Theory is to characterize this integers with properties. For example, you may know that the expression $1 + (1 + 2)$ and the expression $(1 + 1) + 2$ evaluate to the same result. But why? Number Theory explains this and other properties of integers with 6 **axioms**, or ground rules:

The 6 Axioms: $\mathbb{Z}$, the set of integers, is equipped with two operations, " $+$ " and " $\cdot$ ". This means that, for all $a, b \in \mathbb{Z}$ ($\in$ means in), we have $a + b \in \mathbb{Z}, a \cdot b \in \mathbb{Z}$. Moreover, $\mathbb{Z}$ satisfies the following **ring** axioms:

1. 0 is the additive identity, satisfying $a + 0 = 0 + a = a$.
2. 1 is the multiplicative identity, satisfying $1 \cdot a = a \cdot 1 = a$
3. **Commutativity:** $a + b = b + a$, $a \cdot b = b \cdot a$
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. **Distributivity of Multiplication:** $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$
6. **Additive Inverses:** For any a in $\mathbb{Z}$, there exists $-a \in \mathbb{Z}$ such that $a + (-a) = 0$.

You should be familiar with most of the rules - you were probably taught them when you learned arithmetic. The question is how to prove elementary facts. For example, the following:

Example 1: Show that 0, the additive identity, is uniquely defined.

Proof: Assume, for the sake of contradiction, that two elements exist, 0_a and 0_b, such that $0_a \neq 0_b$ and $0_a, 0_b$ satisfy the additive identity property in Axiom 1. Take an element c from $\mathbb{Z}$. From Axiom 1, we have

$$0_a + c = c$$

$$0_b + c = c$$

Hence, we have

$$0_a + c = 0_b + c.$$

Now, we can add the **additive inverse** of c to both sides (defined by Axiom 6) on the right hand side, to get

$$0_a + c + (-c) = 0_b + c + (-c) \iff 0_a + 0 = 0_b + 0 \iff 0_a = 0_b.$$

(The symbol $\iff$ means "if and only if"; in other words, two statements denoted by $\iff$ are equivalent.) But we assumed $0_a \neq 0_b$. Hence, we arrive at a contradiction, and $0_a = 0_b$. Thus, there exists a unique 0. ■

Problem 1. Show that 1, the multiplicative identity, is uniquely defined.

Problem 2. Show the additive inverse of any element $a \in \mathbb{Z}$ is uniquely defined. In other words, show that if a_x, a_y exist with $a + a_x = 0$ and $a + a_y = 0$, then $a_x = a_y$. (The symbol $\in$ means in.)

In the prologue, we saw the example of subtraction being used on both sides of an equation. Now, we'll prove this concretely.

Problem 3. Show that $a + b = a + b'$ implies $b = b'$.

It's also a smart idea to deal with negative numbers, as we are using additive inverses. For example, what's the negative of a negative number? We can frame this using additive inverses.

Example 2. Show that $-(-a) = a$.

Proof: We start from

$$a + (-a) = 0,$$

using Axiom 6. Now, let $a' = -(-a)$ be the additive inverse of $-a$. Adding a' to both sides (possible using Problem 3!) gives

$$a + (-a) + a' = 0 + a' = a' \iff a + ((-a) + a') = a',$$

which follows from Associativity. The term in parenthesis evaluates to 0 from Axiom 6, so we have

$$a + 0 = a' \iff a = a'.$$

Thus, $a = a' = -(-a)$, so $-(-a) = a$ as desired. ■

We've been given the properties of addition and multiplication, but not yet of division and subtraction. How can we translate our axioms into these two operations?

As you might have noticed, subtraction is essentially given to us by Axiom 6. To subtract two numbers a and b, we can add the additive inverse of b to a to get the difference $a - b$. We'll define this formally:

Definition 1. Let $-$ be the **subtraction operator**, where $a - b = a + (-b)$, with $-b$ being the additive inverse of b.

For the next few problems, adapt the definitions of commutative and associative given in the axioms to subtraction. (In other words, substitute $+, \cdot$ signs with $-$ signs.)

Problem 4. Is subtraction commutative? If so, prove it; if not, give a numeric counterexample.

Problem 5. Is subtraction associative? If so, prove it; if not, give a numeric counterexample.

It's more problematic to define division, however, as we're dealing with the integers. We'll come back to this idea very soon.

4 The Order Axioms

We'll assume regular axiomatic properties for addition, subtraction, and multiplication now. In particular, we'll no longer cite facts such as commutativity or associativity. However, it is necessary to cite order axioms for problems in this section.

So far we've seen how to deal with familiar operations such as multiplication, addition and subtraction. But apart from that, our set of integers $\mathbb{Z}$ doesn't have much structure or order.

The Order Axioms. We define $\mathbb{Z}^+$ to be the **set of positive integers**, a subset of $\mathbb{Z}$. $\mathbb{Z}^+$ is equipped with the same operations, $+$ and $\cdot$. It has the following properties:

1. **Additive Closure:** For all $a, b \in \mathbb{Z}^+$, we have $a + b \in \mathbb{Z}^+$.
2. **Multiplicative Closure:** For all $a, b \in \mathbb{Z}^+$, we have $a \cdot b \in \mathbb{Z}^+$.
3. **Nontriviality:** $0 \notin \mathbb{Z}^+$.
4. **Trichotomy:** For all $a \in \mathbb{Z}$, exactly one of the following is true: $a \in \mathbb{Z}^+$, $a = 0$, or $-a \in \mathbb{Z}^+$.

From the axioms alone we can deduce a few things. For example:

Example 3. We have $1 \in \mathbb{Z}^+$ and $-1 \notin \mathbb{Z}^+$.

Proof: Suppose $-1 \in \mathbb{Z}^+$ for contradiction's sake. Then, $(-1)\cdot(-1) = 1 \in \mathbb{Z}^+$ due to multiplicative closure. But note that, from Trichotomy, we cannot have $1 \in \mathbb{Z}^+, -1 \in \mathbb{Z}^+$ both being satisfied. So, we arrive at a contradiction. Since $-1 \neq 0$, from Trichotomy we have $-1 \notin \mathbb{Z}^+$.

Now, from Trichotomy, we have exactly one of $1 \in \mathbb{Z}^+, 1 = 0$, or $-1 \in \mathbb{Z}^+$ being true. Clearly $1 = 0$ is absurd, and we already know that $-1 \notin \mathbb{Z}^+$. Hence, $1 \in \mathbb{Z}^+$.

To conclude, $1 \in \mathbb{Z}^+$ and $-1 \notin \mathbb{Z}^+$. ■

Problem 6. Show that $2 \in \mathbb{Z}^+$. (Hint: Use Example 3, and don't overthink it!)

We've now set the basis to order the positive integers.

Definition 2. Define $>, <, \geqslant, \leqslant$ as follows:

- We have $a > b$ if and only if $a - b \in \mathbb{Z}^+$.
- We have $a < b$ if and only if $b - a \in \mathbb{Z}^+$.
- We have $a \geqslant b$ if and only if $a - b \in \mathbb{Z}^+$ or $a - b = 0$.
- We have $a \leqslant b$ if and only if $b - a \in \mathbb{Z}^+$ or $b - a = 0$.

Example 4. $a < b$ and $b < c$ implies $a < c$ for $a, b, c \in \mathbb{Z}^+$.

Proof: We have $b - a \in \mathbb{Z}^+$ and $c - b \in \mathbb{Z}^+$ by definition of $<$. Now, from Additive Closure, note that

$$(c - b) + (b - a) \in \mathbb{Z}^+ \iff c - a \in \mathbb{Z}^+.$$

Thus, $a < c$. ■

Problem 7. Show that $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for $a, b, c \in \mathbb{Z}^+$. (This might require a bit of casework.)

Now we have everything that we need to prove some basic inequality properties.

Problem 8. Suppose $a \leqslant b$ where $a, b \in \mathbb{Z}^+$. Show that, for $c \in \mathbb{Z}^+$, we have $a + c \leqslant b + c$.

Problem 9. Suppose $a \leqslant b$. Show that, for $c \in \mathbb{Z}^+$, we have $a \cdot c \leqslant b \cdot c$.

5 Divisibility

Definition 3. For $a, b \in \mathbb{Z}$, we say that a divides b if there exists $c \in \mathbb{Z}$ such that $ac = b$. We denote this by $a \mid b$. Moreover, we have the extra condition that $a \neq 0$ (to prevent divisibility by 0).

Using this definition, we have a few elementary corollaries.

Example 5. $a \mid a$.

Proof: Note that $a \cdot 1 = a$. Since $1 \in \mathbb{Z}$, we have $a \mid a$. ■

Example 6. For all $a, b, c \in \mathbb{Z}$ with $a \neq 0$, if $a \mid b$, then $a \mid bc$.

Proof: Let k_1 be the integer such that $ak_1 = b$, which exists because $a \mid b$. Then, we have

$$bc = (ak_1)c = a(k_1c).$$

Because k_1c is an integer, it follows that $a \mid bc$. ■

Example 7. For all $a, b, c \in \mathbb{Z}$ with $a \neq 0$, if $a \mid b$, then $a \mid bc$.

Proof: Let k_1 be the integer such that $ak_1 = b$, which exists because $a \mid b$. Then, we have

$$bc = (ak_1)c = a(k_1c).$$

Because k_1c is an integer, it follows that $a \mid bc$. ■

Hence, we finally have all 4 operations. Just as we did with subtraction, we can ask the same questions about associativity and commutativity.

Problem 10. Is division commutative? If so, prove it; if not, give a numeric counterexample.

Problem 11. Is division associative? If so, prove it; if not, give a numeric counterexample.

6 Parting Problems

Problem 12. Suppose $a, b \in \mathbb{Z}^+$. Show that, if $a \mid b$ and $a \geqslant b$, then $a = b$.

Problem 13. Let k be such that $k \mid a, b$ for $k, a, b \in \mathbb{Z}$. Show that $k | (ax + by)$ for all $x, y \in \mathbb{Z}$.

Problem 14. Show, using additive closure, that each element in the sequence $1, 2, 3, \ldots$ is an element of $\mathbb{Z}^+$.

Problem 15. Suppose x is an integer such that, if $x \mid ab$, then either $x \mid a$ or $x \mid b$, for all integers a, b. What properties does x have? (This does not require a proof - rather, correct observations will lead to points.)