Live Round

Middlesex County Academy Math Competition

April 13, 2024

This section of the competition is to be completed by **your team** within **1 hour**.

This section consists of **8 sets of 3 questions each**. You will receive each set only once you hand in the previous set.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

All answers are **positive integers** between 0 and 999 **inclusive**.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Best of luck!

1 Set 1

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1. ____________ 2. ____________ 3. ____________

1. $\triangle ABC$ satisfies $\angle A = 100^\circ$ and $\angle B = 40^\circ$. D is chosen on BC such that AD bisects $\angle A$. Find $\angle CDA$.

2. Danny has stolen all of the donuts from the faculty room! He starts off with x donuts, where x is divisible by 7. However, he then drops two donuts on the floor. Now, the number of donuts he has is divisible by 11. Find the least possible value x.

3. Freddy brought his marbles to school. He gave 35 of the marbles to his friend John, lost 80% of the remaining marbles, and traded 10 of them for a baseball card. He now has only 3% of his original marbles remaining. How many more marbles did Freddy have at the start?

2 Set 2

Names: ____________________

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4. ________ 5. ________ 6. ________

4. Veda is visiting a playground with a roundabout (shaped as a circle) in the middle. Located on the roundabout is a swing set, which is 6 yards west and 8 yards north of the entrance to the playground. The entrance is also located on the roundabout, and a direct path to the swing set from the entrance involves crossing the center of the roundabout. If r is the radius of the roundabout, find r^2.

5. Suppose that $a^2 + ab - 6b^2 = 0$ and $a + b = 5$, where $a, b > 0$. Find the value of $\frac{a}{b}$.

6. Your calculator malfunctions and subtracts 1 from your answer every time you perform an operation. You start with the number 13, and you perform the operations $\times A$, $\div B$, and $+C$ (in that order) for positive digits A, B, and C. The final value is 109. Find the 3 digit number ABC.

3 Set 3

Names: ______________________

Individual IDs: ______________________

Team Name: ______________________

Team ID: ______________________

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7. ________ 8. ________ 9. ________

7. At the MCAMCafe, there are three ice cream flavors to choose from: vanilla, chocolate, and strawberry. The customer wants a cone with three scoops of ice cream. The customer does NOT want:

   1) "Chocolate and strawberry to touch."
   2) "Vanilla on the top or bottom."
   3) "More than one chocolate scoop."
   4) "Strawberry and vanilla to touch."

   If vanilla = 1, chocolate = 2, and strawberry = 3, what is the order of the scoops from top to bottom? (Ex: if vanilla was on top, chocolate in the middle, and strawberry on the bottom, the answer would be 123). Duplicate flavors are allowed.

8. An 8-sided die with faces ranging from 1-8 is biased such that the probability of rolling an even number is three times as likely as rolling an odd number. The expected value of the face showing up after 1 roll can be expressed as $\frac{A}{B}$, where A, B are positive and $\frac{A}{B}$ is in lowest terms. Find $A + B$.

9. Consider the equation $x^3 + 3x^2 - 2x - 6 = 0$, where p, q, and r are three distinct real solutions to the equation. Compute $(p^2 + q^2 + r^2)^2$.

4 Set 4

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

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10. ____________ 11. ____________ 12. ____________

10. The GCF (Greatest Common Factor) of 1001 and a three-digit number x is 91. The GCF of 1001 and $x+1$ is 11. Find x.

11. A number is called reversible if it is NOT a palindrome and shares at least two distinct prime factors with the number made by reversing its digits. For example, 24 is reversible because it shares at least two common prime factors with its reverse, 42. Find the sum of all two-digit reversible numbers.

12. Observe the following equation: $\sqrt{13|a|-29} = \sqrt{a^2+13}$. If x is the sum of the possible solutions to this equation, find $\frac{x^2}{13}$.

5 Set 5

Names: ____________________

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Team Name: ____________________

Team ID: ____________________

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13. __________ 14. __________ 15. __________

13. Mickey Mouse is standing at the bottom left of a 6 inch x 8 inch grid. He can never leave the grid and always moves either one inch to the right or one inch up. He wishes to rescue Minnie Mouse, who is standing at the top right of this grid. If he can achieve this in n ways, find the remainder when n is divided by 1000.

14. Edgar begins from point A and walks 14 units to point B. He then walks along a 135° arc around the circle centered at A going through B, taking him to a new point C. The area of this triangle ABC can be expressed as the form $m\sqrt{n}$, for positive integers m and n where n is not divisible by the square of any prime. Find $m + n$.

15. You are an aspiring lemonade vendor who wishes to sell lemonade in whole-number quantities from 1 cup to 40 cups. However, due to the current economic crisis, you cannot buy measuring cups and can only buy 4 unlabeled jars to draw and sell lemonade. You need to sell your lemonade using exact measurements, otherwise your customers will not be happy. Moreover, you want your 4 jars to be able to contain all 40 cups, for transportation purposes. What should the size of the largest jar be, in cups, in order for the 4 jars to measure every whole value from 1 to 40?

6 Set 6

Names: ____________________

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Team ID: ____________________

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16. __________ 17. __________ 18. __________

16. Let $S = \{1, 2, 3, 4, 5\}$. Find the number of ordered pairs (S_1, S_2) of subsets of S such that $|S_1 \cap S_2| = 2$.

    (Subsets of a larger set are just groups of elements from that set; for example, $A = \{1, 3, 4\}$ is a subset of S, but $B = \{1, 6\}$ is not. The notation $S_1 \cap S_2$ signifies the set that is the intersection of two sets, i.e. the set of all elements common to both sets, and $|A|$ is the number of elements in a set A.)

17. 6 men and 6 women sit around a circular table. Let x be the number of ways there are to arrange them if every man must sit next to 2 women and every woman must sit next to 2 men. Find the remainder when x is divided by 1000.

18. Compute the remainder when the sum of the first 12 positive integer cubes (i.e. $1^3 + \cdots + 12^3$) is divided by 1000.

7 Set 7

Names: ______________________________

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19. ____________ 20. ____________ 21. ____________

19. A coin has a diameter of 0.5 inches. It is flipped and lands onto an infinitely large checkerboard where each square has a side length of 1 inch. If there is a probability $\frac{P}{Q}$ (where $\frac{P}{Q}$ is in reduced form and P, Q are positive integers) that the coin, upon landing, only touches the interior of a single square, what is the value of $P + Q$?

20. Find the value of a for which the infinite sum

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{x^n}$$

is equal to $\frac{1}{x-a}$.

21. Let θ be the minimal positive solution to the equation

$$4\cos^2\theta - 3 = 2\sin\theta.$$

If $\theta = \frac{\pi}{a}$ for some integer a, find a.

8 Set 8

Names: ____________________

Individual IDs: ____________________

Team Name: ____________________

Team ID: ____________________

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22. ____________ 23. ____________ 24. ____________

22. Let $\triangle ABC$ be such that $AB = 3$, $BC = 5$, and $CA = 7$. Let R be the circumradius of $\triangle ABC$, and let r be the inradius of $\triangle ABC$. Find $\frac{R^2}{r^2}$. (The circumradius of a triangle is the radius of the circle going through all vertices of the triangle, and the inradius of a triangle is the radius of the circle inscribed inside the triangle tangent to all 3 of it's sides.)

23. Let f be a function such that

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x^2 + 3.$$

Find $|9f(3)|$.

24. Let $S = \{1, 2, 3, 4, 5, 6\}$ be a set. Find the number of pairs of sets (possibly empty or nondistinct) (S_1, S_2) that are disjoint, i.e. do not share any common elements, where S_1, S_2 are subsets of S.