Team Round

Middlesex County Academy

March 18, 2023

This section of the competition is to be completed by **your team** within **1 hour**. This section consists of **20 questions**.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

Answers must be **exact** (do not approximate π) and in **simplest form**, with all fractions expressed as improper fractions.

Examples of **unacceptable answers** include: $\frac{4}{6}$, $1\frac{1}{3}$, $3+2$.

Examples of **acceptable answers** include: $\frac{2}{3}$, $\frac{4}{3}$, 5.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Either **exact decimal answers** or **improper fractions** will be accepted (i.e. 0.25 and $\frac{1}{4}$ are both acceptable).

Some questions will require a brief explanation. Additionally, questions may have no answer. If so, the correct response is **"None"**. Problems that are not proof-based do not need an explanation.

A note on proofs: The proofs in this team round are designed so that **you will not need more than 3 or** 4 **sentences**. A good explanation should be brief, to the point, and easy to follow.

Best of luck!

1 Answer Sheet

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

10. ______________________________

11. ______________________________

12. ______________________________

13. ______________________________

14. ______________________________

15. ______________________________

16. ______________________________

17. ______________________________

18. ______________________________

19. ______________________________

20. ______________________________

2 Set Notation

The origins of **set theory** come from a paper in 1874 written by Georg Cantor, a German mathematician. Since then set theory has taken on many different forms, allowing us to better understand the concepts of infinity and special areas in computer science.

To start our exploration of sets, consider the following notations first.

Proposition 1.1. Adopt the following notations:

i) A **set** is closed using curly braces ("{" and "}") and contains elements in between. It is generally denoted with a capital letter. For example, the set $A = \{1, 2, 3, 5\}$ is a valid set, but **not** $A = [1, 2, 3, 5]$. Notice that set **does not** contain repeats of elements; so, for example, $A = \{1, 1, 2, 3, 5\}$ is not a valid set.

(ii) The symbol $\varnothing$ is used to denote the **empty set**, a set without any elements.

(iii) The **universal set** U is the set of all elements that are being considered for the problem. Common universal sets include $\mathbb{Z}^+$ (the set of all positive integers) and $\mathbb{R}$ (the set of all real numbers).

(iv) The **cardinality** of a set is the number of elements in the set. Usually, it is denoted as either $\#(A)$ or $|A|$. The second notation will be used during this round.

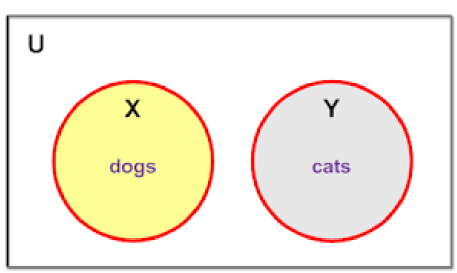
(v) The **complement** of a set A, denoted as A', is the set of all elements not in the set A that are in the universal set U. For example, for the set $A = \{2, 3, 4, \dots\}$ $A' = \{1\}$ when the universal set is $\mathbb{Z}^+$.

(vi) A set A is a **subset** of another set B if all elements of A are also elements of B. If A is a subset of B, it is denoted as $A \subseteq B$. (For simplicity purposes, if $A = B$, then the statements $A \subseteq B$ and $B \subseteq A$ are true.)

(vii) The **union** of two sets A and B is the set of all elements in A and B, included. If repetitions exist, these elements are deleted. The union is denoted with the same $A \cup B$. For example, the union of the sets $A = \{1, 2\}$ and $B = \{2, 3\}$ is $\{1, 2, 3\}$.

(viii) The **intersection** of two sets A and B is the set of all elements **common to** A and B (i.e all elements in both sets). The intersection is denoted with $A \cap B$. For example, the intersection of the sets $A = \{1, 2\}$ and $B = \{2, 3\}$ is $\{2\}$.

As an example, here's a visual demonstration of sets; here, dogs and cats are individual sets, and the universal set is the set of all animals. A subset of the set of all animals would be the set of all pets.



Problem 1: Let $A = \{1, 4, 5, 7, 9\}, B = \{4, 6, 7, 8\}, C = \{1, 2, 3, 4, 5\}$. Find each of the following:

(a) $A \cup B$

(b) $B \cap C$

(c) $C \cup (A \cap B)$

(d) $A \cap (C \cup B)$

(e) $A \cap A$

(f) $A \cup (A \cup (B \cup C))$

Problem 2: Answer the following questions with true or false:

(a) Elements in at least one of two sets A and B are also in their intersection.

(b) The universal set of elements is a subset of all sets containing elements belonging to the universal set.

(c) $|A \cup B| + |A \cap B| = |A| + |B|$. If this is true, prove it; otherwise, provide a counterexample.

Problem 3: Prove or disprove the following:

The intersection of a set and it's complement is $\varnothing$.

Problem 4: Prove or disprove: $A \cap (B \cap C) = (A \cap B) \cap C$ for arbitrary sets A, B, C.

Hint: Consider an arbitrary element of either A, B, C; when does it appear in $A \cap (B \cap C)$, and when does it appear in $(A \cap B) \cap C$?

Problem 5: Find the number of 8 element subsets S of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that the average of the elements in S is 6.

3 Principle of Inclusion & Exclusion

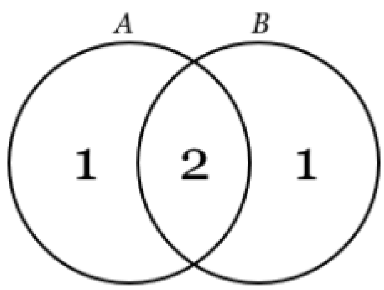
The **Principle of Inclusion-Exclusion** (abbreviated PIE) provides an organized method/formula to find the number of elements in the union of a given group of sets, the size of each set, and the size of all possible intersections among the sets.

Two Set Example:

Assume we are given the sizes of two sets, $|A_1|$ and $|A_2|$, and the size of their intersection, $|A_1 \cap A_2|$. We wish to find the size of their union, $|A_1 \cup A_2|$.

To find the union, we can add $|A_1|$ and $|A_2|$. In doing so, we know we have counted everything in $|A_1 \cup A_2|$ at least once. However, some things were counted twice. The elements that were counted twice are precisely those in $A_1 \cap A_2$. Thus, we have that:

$$\boxed{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|}$$



Here's a visual representation of the Two-set example using **venn diagrams**. The region marked 2 is the intersection of sets A and B, or $A \cap B$. Notice that it is counted once by A and once by B, for a total of 2 times. Thus, to find $|A \cup B|$, we must subtract $|A \cap B|$ once from $|A| + |B|$, hence the formula above.

Problem 6: If there are a total of 8 elements (not necessarily distinct) in A and B, and there are 2 elements common to A and B, find $|A \cup B|$.

Problem 7: If 10 girls are on my school's soccer team, 15 girls are on my school's basketball team, and 6 girls play both sports, then how many girls play soccer or basketball?

Problem 8: At my school, the only foreign languages offered are Spanish and Hindi, and there are 30 students enrolled in at least one of the classes. If 22 students are in the Spanish class and 17 students are in the Hindi class, then how many students are taking both languages?

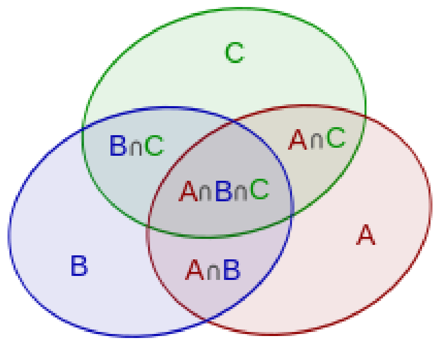
Three Set Example:

Assume we are given the sizes of three sets, $|A_1|, |A_2|,$ and $|A_3|$, the size of their pairwise intersections, $|A_1 \cap A_2|, |A_2 \cap A_3|$, and $|A_3 \cap A_1|$, and the size their overall intersection, $|A_1 \cap A_2 \cap A_3|$. We wish to find the size of their union, $|A_1 \cup A_2 \cup A_3|$.

Just like in the Two Set Example, we start with the sum of the sizes of the individual sets $|A_1| + |A_2| + |A_3|$. We have counted the elements in exactly one of the original three sets once, but we've obviously counted other things twice, and even thrice! To account for the elements that are in two of the three sets, we first subtract out $|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|$.

Now we have correctly accounted for them since we counted them twice originally, and just subtracted them out once. However, the elements that are in all three sets were originally counted three times and then subtracted out three times. We have to add back in $|A_1 \cap A_2 \cap A_3|$. Putting this all together gives:

$$\boxed{|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|}$$



Above is a visual demonstration of the 3 set example using venn diagrams. The union $|A \cup B \cup C|$ is simply the entire region formed by the 3 circles. With this diagram, we can see that $A \cap B$ is counted twice in A and B; hence it must be subtracted once, as in the formula, for it to be counted once. Moreover, $|A \cap B \cap C|$ is counted 3 times in $|A|, |B|, |C|$, and has also been subtracted 3 times in $|A \cap B|, |B \cap C|, |C \cap A|$; thus, it has to be added back once.

Now, try applying what you have learned to more difficult problems.

Problem 9: Suppose we are looking at a different school that offers 5 languages: Spanish, Hindi, French, Latin, and Mandarin. The following is a list of how many students take each language or group of languages: 308 take Spanish, 291 take Hindi, 564 take French, 111 take Latin, 299 take Mandarin, 188 take Spanish and Hindi, 245 take Spanish and French, 36 take Spanish and Latin, 176 take Spanish and Mandarin, 217 take Hindi and French, 70 take Hindi and Latin, 205 take Hindi and Mandarin, 55 take French and Latin, 142 take French and Mandarin, 23 take Latin and Mandarin, 45 take Spanish, Hindi, and French, 46 take Spanish, Hindi, and Latin, 47 take Spanish, Hindi, and Mandarin, 47 take Spanish, French, and Latin, 48 take Spanish, French, and Mandarin, 49 take Spanish, Latin, and Mandarin, 50 take Hindi, French, and Latin, 51 take Hindi, French, and Mandarin, 52 take Hindi, Latin, and Mandarin, 53 take French, Latin, and Mandarin, 23 take Spanish, Hindi, French, and Latin, 22 take Spanish, Hindi, French, and Mandarin, 21 take Spanish, Hindi, Latin, and Mandarin, 19 take Spanish, French, Latin, and Mandarin, 18 take Hindi, French, Latin, and Mandarin, 7 students take all 5 languages, and 1 student takes none of the languages. (Phew, that was long!) **How many students are in the school?**

Problem 10: 6 mathletes are sitting in a row of 6 chairs to take the MCAMC Team Round. Find the number of ways they may be seated such that no 3 adjacent mathletes are in increasing order of height, from left to right. You may assume that you are standing in front of the mathletes such that they are facing you and that no 2 mathletes have the same height.

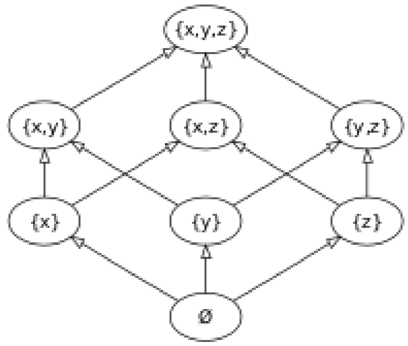
For the curious. The general form of PIE can be written as follows:

$$
\begin{aligned}
|A_1 \cup A_2 \cdots \cup A_N| = & \sum_{1 \leqslant i \leqslant n} |A_i| \\
& - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| \\
& + \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| \\
& \vdots \\
& + (-1)^{m+1} \sum |A_{a_1} \cap A_{a_2} \cdots \cap A_{a_m}| \\
& \vdots \\
& + (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|.
\end{aligned}
$$

4 Power Set

Now that we've talked significantly about sets, we introduce the notion of a **power set**.

Definition. A **power set** of a set A is the set of all subsets of A. It is denoted as $P(A)$.



Example. The power set of $A = \{1, 2\}$ is $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

The power set has some interesting properties. For example;

Proposition 3.1: Properties of the power set.

- The empty set is always an element of the power set.
- $A \subseteq P(A)$.
- $A \subseteq B$ implies $P(A) \subseteq P(B)$.
- $A \cap B = \varnothing$ implies $P(A) \cap P(B) = \varnothing$.

Let's prove one of these first.

Problem 11: Prove the final property: $A \cap B = \varnothing$ implies $P(A) \cap P(B) = \varnothing$.

The idea of a power set is applicable to many areas of mathematics outside of set theory. See if you can solve the following problem and relate it to the notion of a power set. (You'll need this idea soon.)

Problem 12: You are at an ice cream parlor that offers you scoops of chocolate, strawberry, mint, vanilla, and butter pecan. If you can choose any number of scoops on your cone, and the order of the flavors on your cone does not matter, how many different types of ice cream cones can you get? (You can't get just an ice cream cone, though; that's not cool.)

Now, we'll tackle a more general problem.

Problem 13: Given $|A| = n$, find $|P(A)|$, and prove your assertion.

5 Cartesian Product

The **Cartesian Product** of 2 sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B. It is named after René Descartes, whose formulation of analytic geometry gave rise to the concept.

In terms of set-builder notation, the Cartesian product can be expressed as:

$A \times B = \{(a, b) \mid a \in A \textbf{ and } b \in B\}$

Problem 14: If set A is defined as $\{x, y, z\}$ and set B is defined as $\{1, 2, 3\}$, find the set $A \times B$.

Problem 15: Prove or disprove that $A \times B$ is commutative (i.e. $A \times B = B \times A$). If this isn't true, provide sets A, B for which this doesn't hold.

Problem 16:

(a) Show that the Cartesian Product is not associative (i.e. $(A \times B) \times C \neq A \times (B \times C)$).

(b) Make a generalization about when the Cartesian product of 3 sets **would** be associative. (This should not be more than 1 sentence long.)

6 Parting Problems

Problem 17: Let A be the set of factors of the number 3003. Find $|A|$.

Problem 18: There are 30 students enrolled at the Edison Academy Magnet School, a highly selective school located in the state of New Jersey. Of them, 20 play soccer, 18 play football, and 13 play baseball. 10 of them play both soccer and football, 7 of them play both football and baseball, and 6 of them play soccer and baseball. Given that all students play some sport, find the number of people that play exactly two sports.

Problem 19: Define A/B as the set of elements belonging to B not belonging to A. Define $A \odot B$ as $(A/B) \cap (B/A)$. Are there any interesting properties about $\odot$? Experiment with small sets A and B, and report your observations.

Problem 20. Andy and Andrew, Brian and Ben, and Carly and Catherine are 3 sets of twins who go to the same school. Today is picture day, and the 6 are called down for a special twin portrait photo. They sit in a row of 6 chairs. The photographers, however, do not want someone to sit next to their twin, fearing that it might lead to bickering over who is the more photogenic twin. Find the number of ways the photographers can seat the students so that a quarrel doesn't arise.

7 Solutions

1. (a) $\{1, 4, 5, 6, 7, 8, 9\}$

   (b) $\{4\}$

   (c) $\{1, 2, 3, 4, 5, 7\}$

   (d) $\{1, 4, 5, 7\}$

   (e) $\{1, 4, 5, 7, 9\}$

   (f) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

2. (a) **False:** This is only true if an element is present in both sets A and B.

   (b) **False:** The opposite is true - all sets made of elements from the universal set are subsets of the universal set.

   (c) **True:** Consider an element x in A. If it is not in B, it is counted once in $A \cup B$ on the left-hand side. If it is in B, it is counted twice on the right-hand side ($|A|$ and $|B|$) and twice on the left-hand side ($|A \cup B|$ and $|A \cap B|$). Thus, all elements are counted an equal number of times in both sets, implying that the two sides of the equation are the same.

3. This is **True**. For an element in the set A, by the definition of the complement, it is not present in the complement. Thus, it is not present in the intersection of the two sets. So, the set of intersections is the empty set.

4. This is in fact **True**. Notice that an element is only present on the right-hand side if it is an element of A and $B \cap C$, which is simply the set of all elements in all of A, B, C; the same applies on the left-hand side, and so the two sets are equal.

5. Notice that a subset of 8 elements with an average of 6 must have a total sum of $8 \cdot 6 = 48$. Since the sum of all elements is $1 + 2 + \cdots + 10 = 55$, it follows that two elements with a sum of $55 - 48 = 7$ must be excluded. This is achieved by either $\{1, 6\}$, $\{2, 5\}$ and $\{3, 4\}$, so the answer is $\boxed{3}$.

6. From PIE, we conclude that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = (8) - (2) = \boxed{6}.$$

7. Consider the set A as the set of all basketball players and the set B as the set of all soccer players. We find

$$|A \cup B| = |A| + |B| - |A \cap B| = 15 + 10 - 6 = \boxed{19} \text{ girls.}$$

8. In this case, let A be the set of people taking Spanish and B be the set of people taking Hindi. We rearrange to get

$$|A \cap B| = |A| + |B| - |A \cup B| = 22 + 17 - 30 = \boxed{9} \text{ students.}$$

9. First, discard the 1 student taking no languages for now; we will consider it later. Using the multi-number form of PIE, our answer becomes

$$(308 + 291 + 564 + 111 + 299) - (188 + 245 + 36 + 176 + 217 + 70 + 205 + 55 + 142 + 23)$$

$$+(45 + 46 + 47 + 47 + 48 + 49 + 50 + 51 + 52 + 53) - (23 + 22 + 21 + 19 + 18) + 5.$$

   After crunching all the numbers together, we get 606 from this expression. But we have to add back the one student that took no languages, yielding an answer of $\boxed{607}$ students.

10. There are 720 ways to order all the mathletes. There are 120 ways for at least 3 people to be sitting in order of increasing height and 4 ways for those 3 people to be sitting adjacent to one another so we subtract 480. Then we add 122 back ways for any 2 gruops of people to be ordered in increasing height. Then we subtract 14 ways for any 3 groups of people to be increasing height order. Finally, we add back 1 way if the entire row is in ordered height. Thus, we get $720 - 480 + 122 - 14 + 1 = \boxed{349}$.

11. If no elements are common to A and B, then this implies that, if x is an element in A, then it is not an element of B. Thus, all sets containing x as an element in $P(A)$ are not in $P(B)$. Thus, their intersection is also empty.

12. This is just the size of a power set with 5 elements which is equal to $2^5 = 32$. However, we subtract 1 since we can't have an empty cone giving us the answer of $\boxed{31}$.

13. $\boxed{|P(A)| = 2^n}$ The proofs may vary but it can boil down to the fact that for every element in the original set when making the power set, you can choose to either include it or not include it. This means that there are 2 choices for each of the n elements, leading to 2^n as the cardinality of the power set.

14. Based on the formula for a cartesian product you get $\{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3),(z,1),(z,2),(z,3)\}$.

15. Counter example: $A \times B$ where $A = \{1,2\}$ and $B = \{3,4\}$. In this case, the ordered pairs would be in reverse order. Any example where A is not equal to B and neither A nor B is the empty set will work.

16. Counter example: $A = B = C = 1$. $(A \times A) \times A = \{((1,1),1)\} \neq (1,(1,1)) = A \times (A \times A)$. Any example where none of the sets are empty will work.

    The Cartesian product will be associative when one of the sets is $\boxed{\text{empty}}$.

17. Notice that $3003 = 3 \times 7 \times 11 \times 13$. Now, consider the power set of $A = \{3,7,11,13\}$; notice that for every element in $P(A)$, the product of the elements forms a factor of 3003. Thus, our answer is $|P(A)| = 2^4 = \boxed{16}$.

18. First, we calculate the number of people who play all 3 sports. We have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| + |A \cap B \cap C| \\ &\rightarrow |A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| \\ &= 30 - 20 - 18 - 13 + 10 + 7 + 6 = 2. \end{aligned}$$

    Now, to find the number of people playing exactly two sports, notice that the number of people playing all 3 sports is counted 3 times in the count of people playing two sports; so, we must subtract it 3 times, to get $10 + 7 + 6 - 3(2) = \boxed{17}$.

19. We claim that $A \odot B = \varnothing$ regardless of the elements in A and B. Notice that all elements in A/B are elements of B that aren't elements of set A, and all elements in B/A are elements of set A not in B. But this implies that no elements are common between the two sets, as otherwise, an element not in A would be in A, a contradiction. Thus, the intersection of these two sets is $\varnothing$.

20. We present two approaches, both of which are completely valid.

    Solution 1: We use PIE and complementary counting. Suppose that some pair of twins sits next to each other; then there are 5 ways to choose where they sit next to each other, 2 ways to order them, and 4! ways to seat the rest of the students. For all 3 pairs of twins, we get $3 \times 5 \times 2 \times 4! = 30 \times 4! = 720$.

    Now, suppose that 2 pairs of twins sit next to each other. There are 3 ways to choose which 2 pairs sit next to each other. There are a total of $3 + 2 + 2 + 2 + 3 = 12$ ways to seat both pairs, based on casework. There are then $(2!)^2$ ways to order these two pairs of twins, and 2! ways to seat the last two students. In total, there are $3 \times 12 \times (2!)^3 = 3 \times 12 \times 8 = 288$ ways to seat the students.

    Now, consider the final case where all 3 pairs of twins set next to each other. There are 3! ways to order the pairs of twins, and $(2!)^3$ ways to seat within pairs of twins. There are $3! \cdot (2!)^3 = 48$ ways in this case.

From PIE, there are thus $720 - 288 + 48 = 480$ invalid seatings. There are a total of $6!$ ways to seat the students, giving us a final answer of $6! - 480 = \boxed{240}$.

Solution 2: We use casework.

There are 6 ways to choose who sits in the first seat. Then, their twin can sit in 4 possible positions.

If their twin sits in the 3rd position, the other two sets of twins can sit in $4 \times 2 = 8$ ways.

If their twin sits in the 4th position, the other two sets of twins can sit in $4 \times 2 \times 2 = 16$ ways.

If their twin sits in the 5th position, this is practically identical to the case where the twin sits in the 3rd position, for a total of 8 ways.

If their twin sits in the 6th position, there are $4 \times 2 = 8$ ways to seat the rest of the students.

In total, our answer is $6 \times (8 + 16 + 8 + 8) = \boxed{240}$ ways.