Live Round

Middlesex County Academy

March 18, 2023

This section of the competition is to be completed by **your team** within **1 hour**.

This section consists of **10 sets of 3 questions each**. You will receive each set only once you hand in the previous set.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

All answers are **positive integers** between 0 and 999 **inclusive**.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Best of luck!

1 Set 1

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1. ____________ 2. ____________ 3. ____________

1. An equilateral triangle has side length 12. The area of the triangle can be written as $A\sqrt{B}$. What is $A + B$?
2. There are oranges, apples, and bananas in a bag. There are 25% more oranges than apples and there are 80% more bananas than oranges. If there are 20 oranges, how many fruits are in the bag total?
3. If $x + \frac{1}{x} = 4$, then what is the value of $x^2 + \frac{1}{x^2}$?

2 Set 2

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

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4. ____________ 5. ____________ 6. ____________

4. Find the sum of the factors of 100.
5. James is 2 miles away from Bob. James walks toward Bob at a rate of 20 yards per minute, and Bob runs toward James at a rate of 10 feet per second. How far away from James' original position, in feet, will the two meet? Assume that both James and Bob are traveling on a straight path. (1 mile = 5280 feet and 1 yard = 3 feet)
6. Define $a\&b = a^2b - b^2a$ and define $a\#b = \frac{a^3}{b} - \frac{b^3}{a}$. The value of $\frac{5\&4}{3\#2}$ can be written as a simplified fraction $\frac{X}{Y}$. What is $X + Y$?

3 Set 3

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

7. ______________ 8. ______________ 9. ______________

7. Four students 1, 2, 3, and 4 are suspected of stealing a pencil from the classroom. They make the following statements when questioned by the teacher:

   1: "It was 3 or 4."

   2: "It wasn't 3 or me."

   3: "1 is lying."

   4: "2 is saying the truth."

   If the teacher knows only one of them is truthful, find the sum of the numbers of all remaining suspects.

8. A regular tetrahedron is inscribed in a cube which is inscribed in a sphere of radius 2. The square of the volume of the regular tetrahedron can be written as $\frac{A}{B}$. What is $A + B$ mod 1000?

9. If $\tan(x) + \tan(y) = 20$, and $\tan^2 x + \tan^2 y = 418$, find $\tan(x + y)$.

4 Set 4

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

10. ____________ 11. ____________ 12. ____________

10. The sum of the common logarithms of each factor of 200000 can be written as $A\log(B) + C$. Find $A + B + C$.
11. Triangle ABC has $AB = 5$, $BC = 7$, and point E on AC. $AE = 5$ and $EC = 3$. What is the square of the area of triangle ABC?
12. Having resolved their conflict, two of the students decide to play a game of cards. They find that one unknown card is missing from their deck, but each go ahead and draw a card anyway without replacement. If the probability that both of them draw cards from the same suit can be written as a simplified fraction $\frac{A}{B}$, what is $A + B$?

5 Set 5

Names: ________________________________

Individual IDs: ____________________________

Team Name: ______________________________

Team ID: ________________________________

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13. __________ 14. __________ 15. __________

13. Jack is generating a list of numbers starting with 2023. He rolls a die to choose the next number. If it lands on an even number, he doubles the previous number and adds 1. If it lands on an odd number, he halves the previous number and adds 1. The probability that the 4th term of Jack's sequence is an integer can be written as a simplified fraction $\frac{A}{B}$. What is $A + B$?

14. There are 10 library books in a specific order on a shelf. All of the books are removed from the shelf and randomly put back. The probability that exactly 7 of the books are returned to their original positions can be expressed as a simplified fraction $\frac{A}{B}$. What is $A + B$ mod 1000?

15. Point A is located at $(2, 4)$. Point A is reflected across the x-axis, reflected across the line $y = x$, scaled by a factor of 3, rotated 90 degrees counterclockwise about the origin, and translated up 4 units and right 4 units, in that order. This point is labeled point B. The midpoint is then found between point A and point B. This new point is then translated vertically onto the line $y = -x$ and is then labeled point C. What is the area of ΔABC?

6 Set 6

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

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16. ____________ 17. ____________ 18. ____________

16. If $x = 2^8 3^{12}$, find the number of positive integer factors of x^3 that don't divide x.
17. Four runners can go around a track in 3, 4, 5, and 6 minutes respectively. Let x represent the first time, in minutes after they start running, that any two runners meet at the starting mark together. Let y represent the first time that any three meet at the starting mark, and z represent when all four meet at the start. Find $x + y + z$.
18. A set of 3 fair six-sided die numbered 1 through 6 are rolled 3 times. If the probability that the product of the numbers rolled is divisible by 3 can be rewritten as $\frac{A}{B}$ for relatively prime positive integers A, B, find $A + B$.

7 Set 7

Names: ____________________

Individual IDs: ____________________

Team Name: ____________________

Team ID: ____________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

19. ________ 20. ________ 21. ________

19. Circles with radii 2, 4, and 6, are mutually externally tangent. The area of the triangle formed by connecting the 3 points of tangency can be expressed as a simplified fraction $\frac{A}{B}$. What is $A + B$?
20. In the 26 term sequence A, B, C, ... Y, Z, the value of C is 8 and the sum of every 3 consecutive terms is 2023. What is $A + Z$ mod 1000?
21. The digits of a three digit positive integer $\underline{A}\ \underline{B}\ \underline{C}$ can be rearranged to make another three digit integer $\underline{C}\ \underline{A}\ \underline{B}$. Find the mean of all possible values for ABC if ABC - CAB = 198.

8 Set 8

Names: ______________________________

Individual IDs: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

22. ____________ 23. ____________ 24. ____________

22. If $log_{2\sqrt{2}cos(x)} sin^2(x) = 2$, the positive solution for $cot(x)$ can be expressed as a rationalized fraction $\frac{\sqrt{A}}{B}$. Find $A + B$.
23. For what value of n does $49i - 50 = ni^n + (n-1)i^{(n-1)} + ... + 2i^2 + i$?
24. There is a very peculiar clock (numbered 1 to 12) that doesn't really keep time. It has only one hand that starts off in the 1 position. After 1 hour, the hand moves to the next number - 2. After the next hour, the hand moves "forward" 2 to 4. The next hour, it moves forward 4 to 8. The next hour, it moves forward 8 to 4 (wraparound) and so on. In general, at the nth hour, the clock will move 2^{n-1} ticks. To which number will the clock be pointing after 12 hours?

9 Set 9

Names: ____________________

Individual IDs: ____________________

Team Name: ____________________

Team ID: ____________________

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25. __________ 26. __________ 27. __________

25. If we have a function $f(x) = 36x^4 - 36x^3 - x^2 + 9x - 2$ whose roots are x_1, x_2, x_3, and x_4, and the value of $(x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)$ can be expressed as a simplified fraction $\frac{A}{B}$, find $A + B$.
26. If $f(x) = 3x^4 - 7x^3 + 4x^2 + 11x - 3$, the sum of each of the roots squared can be expressed as a simplified fraction in the form $\frac{A}{B}$. Find $A + B$.
27. There are 2023 equally spaced points on a circle. 4 distinct points P, Q, R, and S are chosen. The probability that chord PQ intersects chord RS can be expressed as a simplified fraction $\frac{A}{B}$. What is $A + B$?

10 Set 10

Names: ________________________________

Individual IDs: ________________________________

Team Name: ________________________________

Team ID: ________________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

28. __________ 29. __________ 30. __________

28. Stacy randomly picks 3 distinct numbers between 1 and 7 inclusive and Bryan randomly picks 3 distinct numbers between 1 and 6 inclusive. They both arrange their numbers in decreasing order to form two 3-digit numbers. The probability that Bryan's number is larger than Stacy's number can be expressed as a simplified fraction $\frac{A}{B}$. What is $A + B$?

29. Let ABC be a triangle with $AB = 4, BC = 5, CA = 7$. If the radius of the circle passing through all 3 vertices of the triangle can be expressed as $\frac{A\sqrt{B}}{C}$, where A and C are relatively prime and B is square-free, find $A + B + C$.

30. There are positive numbers x, y, and z that satisfy the equation $x^3y^2z = 288$. What is the minimum value of $2x + y + 3z$?

11 Solutions

1. By the area of an equilateral triangle formula, the answer is $36\sqrt{3}$. Thus, the answer is $\boxed{39}$.

2. If there are 20 oranges then there are 16 apples and 36 bananas. Thus, the total number of fruits is $\boxed{72}$.

3. Square the first equation and subtract 2 from both sides to get the answer of $\boxed{14}$.

4. Finding the factors and summing them gives $\boxed{217}$.

5. Notice that, every second, Phineas and Ferb are 11 feet closer to each other. Thus, they meet after $5280 \cdot 2/11 = 960$ seconds, in which Phineas has run $\boxed{960}$ feet, our answer.

6. Computing the value gets you $\frac{20}{\frac{65}{6}} = \frac{120}{65} = \frac{24}{13}$. Thus, the answer is $\boxed{37}$.

7. Because 2 and 4 agree and only one person is truthful, both of them must be lying. As such, the culprit must be either 2 or 3 (opposite of 2's statement). Because 1 and 3 disagree, it cannot be determined whether 1 is saying the truth or lying. Therefore, no further suspects can be eliminated, making the sum of the remaining suspects' numbers $2 + 3 = \boxed{5}$.

8. The space diagonal of the cube is the diameter of the sphere which is 4. Thus, the side length of the cube is $\frac{4}{\sqrt{3}}$ and its volume is $\frac{4}{3\sqrt{3}}$. Since the volume of a tetrahedron is one third the volume of the cube in which its inscribed, the volume of the tetrahedron is $\frac{4}{3\sqrt{3}}$ and this value squared is $\frac{4096}{243}$. Thus, the answer is $\boxed{339}$.

9. The tangent angle addition formula is $tan(x+y) = \frac{tan(x)+tan(y)}{1-tan(x)tan(y)}$. We already have $tan(x) + tan(y)$, and we can find $tan(x)tan(y)$ by squaring $tan(x)+tan(y)$. Using the given information, we thus find that $tan(x)tan(y) = -9$, and $tan(x+y) = \frac{20}{1-(-9)} = \boxed{2}$.

10. The prime factorization of 200000 is $2^6 5^5$. As such, it has $(6+1)(5+1) = 42$ factors, which can be split into 21 pairs that each multiply to 200000. Using the log addition property, we find that the sum of the common logarithms is simply $21 * log(200000) = 21 * (log(2) + 5) = 21log(2) + 105$. $21 + 2 + 105 = \boxed{128}$.

11. Heron's formula yields $\boxed{300}$.

12. The probability that the first student picks a card from a suit with all 13 cards present is $\frac{39}{51}$. The probability of the second student choosing from the same suit is $\frac{12}{50}$, with 12 cards remaining in the suit and 50 in the deck prior to the second selection. The probability of the first student picking from the suit with a missing card is $\frac{12}{51}$, and the probability of the second student choosing the same suit is $\frac{11}{50}$. The combined probability is thus $\frac{39}{51} * \frac{12}{50} + \frac{12}{51} * \frac{11}{50} = \frac{4}{17}$, and $A + B = 4 + 17 = \boxed{21}$.

13. You can generate all possible sequences till the 4th term and you get that $\frac{1}{2}$ of them are integers. Thus, the answer is $\boxed{3}$.

14. The probability is 10 choose 7 times 2 ways to place the books back correctly divided by the 10! ways to place them back altogether. This yields $\frac{1}{15120}$. Correctly formatted, the answer is $\boxed{121}$.

15. Simple rule-following to get the points $(2, 4)$ for A, $(-2, -8)$ for B, and $(2, -2)$ for C. Then a simple "complete-the-right-triangle" strategy allows to find the area which is $\boxed{12}$

16. Knowing that all factors of x are also factors of x^3, we simply have to find the number of factors of x^3 and subtract the number of factors of x. x has $(8 + 1)(12 + 1) = 117$ factors, and x^3 has $(8 * 3 + 1)(12 * 3 + 1) = 925$ factors. $925 - 117 = \boxed{808}$.

17. The smallest LCM of any two of 3, 4, 5, and 6 is 6. The smallest LCM of any three is 12, and the LCM of all four is 60. $60 + 12 + 6 = \boxed{78}$.

18. Using complementary counting, it suffices to find the probability that a roll of 3 die leads to a product not divisible by 3. The product a single die doesn't have a number divisible by 3 on it's top is 2/3, so our answer is $1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$ for an answer of $\boxed{46}$.

19. The centers of the circles form a 6810 triangle with area 24. Using the sine area formula we can subtract the areas of the 3 smaller triangles on each of the vertices that we don't need. This yields an answer of $\frac{24}{5}$, so $\boxed{29}$.

20. Writing out the first few terms yields a sequence of x, $2015 - x$, 8 which repeats every 3 terms. A and Z have values equal to x and $2015 - x$ respectively which means that their sum is 2015 and the answer is $\boxed{15}$.

21. Writing the first integer as $100A + 10B + C$ and the second as $100C + 10A + B$, the difference becomes $90A + 9B - 99C = 9(10A + B - C) = 198$. Rearranging gives $10A + B = 11(C + 2)$, meaning that the possible values are $331, 442, 553, 664, 775, 886, 997$, the mean of which is $\boxed{664}$.

22. Rearranging the log equation, we get $8cos^2(x) = sin^2(x)$. Dividing both sides by sin^2x and rearranging, we have $cot^2x = \frac{1}{8}$. The positive solution for $cot(x)$ thus simplifies to $\frac{\sqrt{2}}{4}$, and $2 + 4 = \boxed{6}$.

23. Using the repeating pattern of powers of i yields $\boxed{98}$ as the answer.

24. Basically boils down to $(2^{12} - 1 \mod 12) + 1$ which is $\boxed{4}$.

25. From Vieta's we know that the sum of the roots is 1. We can also notice that the polynomial can be factored into $(3x - 1)(3x - 2)(2x - 1)(2x + 1)$. Thus, the roots are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{2}$, and $-\frac{1}{2}$. Rewriting the desired expression in terms of the sum of the roots allows us to find the answer of $\frac{1}{6}$ which we give as $\boxed{7}$.

26. It is known that the sum of the roots of a polynomial is $-\frac{b}{a}$ where a and b are the coefficients of the first and second highest degree terms (including terms with coefficient 0). As such, the sum of the roots is $\frac{7}{3}$. We can then use Newton's Sums to find the sum of each of the roots squared.
$$3P_2 - 7P_1 + 2(4) = 0$$
Plugging in $P_1 = \frac{7}{3}$, we find that $P_2 = \frac{25}{9}$, giving an answer of $25 + 9 = \boxed{34}$.

27. Any 4 points that are picked on the circle can be thought of as a cyclic quadrilateral inscribed in a circle. In this quadrilateral, there are 3 ways to connect the sides out of which only 1 yields intersecting chords. Thus, the probability is $\frac{1}{3}$ and the answer is $\boxed{4}$.

28. The probability that Phineas picks a 7 is $\frac{3}{7}$, in which case Phineas automatically has a bigger number. The probability that Phineas doesn't pick a 7, in which the chance of Phineas' number being larger than Ferb's is equal to the chance of Ferb's number being larger than Phineas', is equal to 1 minus the probability that they pick the same number divided by 2. The probability they pick the same number is $\frac{1}{20}$. $1 - \frac{1}{20} = \frac{19}{20}$ and that divided by 2 is $\frac{19}{40}$. We multiply this value by $\frac{4}{7}$ since we multiply by the probability of not picking a 7 which yields $\frac{76}{280} = \frac{19}{70}$. Adding $\frac{3}{7}$ to $\frac{19}{70}$ yields $\frac{49}{70} = \frac{7}{10}$. The probability that Ferb has a larger number is $1 - \frac{7}{10} = \frac{3}{10}$. Thus, the answer is $\boxed{13}$.

29. From the Extended Law of Sines we find
$$\frac{a}{\sin A} = 2R.$$
The Law of Cosines yields $\cos(A) = -\frac{1}{5}$, implying that $\sin A = \frac{2\sqrt{6}}{5}$. Thus, plugging in the numbers and rationalizing the denominator gives us $\frac{35\sqrt{6}}{24}$ for an answer of $\boxed{65}$.

30. We can break up the desired expression into $(\frac{2x}{3} + \frac{2x}{3} + \frac{2x}{3} + \frac{y}{2} + \frac{y}{2} + 3z)$. Using AM-GM yields the answer of $\boxed{12}$.