MCAMC

Individual Round

Middlesex County Academy

March 18, 2023

This section of the competition is to be completed **individually** within **1 hour**.

This section consists of **30 questions**.

No calculators, notes, compasses, smartphones, smartwatches, or **any other aids** are allowed.

All answers must be written **legibly** on the answer sheet to receive credit.

All answers are **positive integers** between 0 and 999 **inclusive**.

There is **no need to include units** for any answer, and the units are always assumed to be the units in the question.

Best of luck!

1 Answer Sheet

Name: ______________________________

Individual ID: ______________________________

Team Name: ______________________________

Team ID: ______________________________

Please write your answers on this sheet **legibly**. Follow the rules outlined on the first page.

1. ________	11. ________	21. ________
2. ________	12. ________	22. ________
3. ________	13. ________	23. ________
4. ________	14. ________	24. ________
5. ________	15. ________	25. ________
6. ________	16. ________	26. ________
7. ________	17. ________	27. ________
8. ________	18. ________	28. ________
9. ________	19. ________	29. ________
10. ________	20. ________	30. ________

2 Problems

1. Find the fifth term of a geometric sequence where the first term is 3 and the common ratio is 3.

2. The base of isosceles $\triangle ABC$ is 16 and its area is 48. What is its height?

3. The mean of the 5 integers 4, 10, 12, 16, and x is 20. What is the median of the 5 numbers?

4. You have a standard deck of 52 cards. The probability that the first card you pick is the 10 of hearts or the ace of spades can be expressed as a fraction $\frac{A}{B}$. What is $A + B$?

5. There are 5 lines in a plane. Each line intersects every other line. How many points of intersection are there?

6. How many numbers under 100 are divisible by 2 or 3, but not divisible by 4?

7. Suppose you flip a fair coin 8 times in a row. The probability that it lands heads on the second attempt and tails on the seventh attempt can be expressed as a fraction $\frac{A}{B}$. What is $A + B$?

8. An integer x is between 10 and 20 inclusive. How many integers can be the geometric mean of 8 and x?

9. 4 friends are talking to each other: Bob, Leroy, Sam, and Calvin. Each of these friends is either from the wizard clan or the pirate clan. Wizards always tell the truth and pirates always lie. The 4 friends make the following 4 statements:

   Leroy: "Bob and I are not from the same clan."

   Sam: "Calvin is a pirate."

   Calvin: "Sam is a pirate."

   Bob: "Out of all of us, at least 2 are wizards."

   How many of the friends are wizards?

10. A rectangular prism has side lengths 4 inches by 5 inches by 6 inches. A 3-inch by 3-inch square hole is cut into the center of each face, parallel to the edges of the rectangular prism, and goes all the way through the prism. What is the volume, in cubic inches, or the remaining rectangular prism?

11. You are gifted a magical card from a member of the wizard clan which displays a letter on each side (M, C, A).

    Every time you flip the card, the letter randomly changes. If the letters cannot change to themselves and the starting letter is M, what the BEST chance that after five flips, the ending letter is C can be expressed as a/b, where a and b are relatively prime positive integers. What is a+b?

12. The number of palindromes (numbers that are the same when both read forwards and backwards) between 0 and $10^{10} - 1$ can be written in the form $a \times 10^n - k$. Where $a, n,$ and k are positive integers. Find the sum $a + n + k$.

13. The letters M, C, A, M, C are cycled 2022 times, creating a list with 2023 orderings of the letters, as shown below.

$$MCAMC$$

$$CAMCM$$

$$...$$

$$AMCMC$$

How many of these orderings start with the letter C?

14. The polynomial $x^2 - ax + 2023$ has 2 positive integer roots. What is the smallest possible value of a?

15. Observe the following equation: $\sqrt{3|x| + 4} = \sqrt{x^2 - 6}$. What is the product of the possible solutions to this problem raised to the 2nd power?

16. How many integers x satisfy the following inequality?

$$log_2 x \geqslant 0.5x$$

17. A set of 1234 cards are laid face down in a row and are numbered from 1 to 1234 from left to right in that order. The dealer plans to flip cards in multiple steps from left to right. Starting with the card labeled 1, the dealer first flips every 4th card (1, 5, 9... 1233), after which he flips every 3rd card starting with the card labeled 2. He then flips every other card starting with the card labeled 3, and finally he flips every single card starting with the card labeled 4. How many cards are left face up at the end?

18. Find the sum of the products of x and y in all ordered pairs of positive integers (x, y) that satisfy the equation $x^2y - x^2 - 3y + 4 = 14$.

19. Find the absolute value of $(1 - 2) + (2 - 4) + (3 - 6) + ... + (2023 - 4046)$, mod 1000.

20. In how many distinct ways can you organize the letters in $MCAMC$?

21. Equilateral triangle ABC has sides of length 4. Triangle DEF is formed by connecting the midpoints of he sides of triangle ABC. What is the square of the area of triangle DEF?

22. x is the largest integer for which the product of -120 and x is a perfect square, and y is the smallest positive integer for which the product of -120 and y is a perfect cube. Note that negative numbers cannot be considered perfect squares. Find $|x + y|$.

23. If $log(x^2) = a$ and $log(x^b) = a^2b$, and a can be expressed as a simplified fraction, what is the sum of the numerator and denominator of a?

24. How many positive integers n are there such that $1000n$ is divisible by the sum of the first n positive odd integers?

25. An orange and 3 apples are worth 17 dollars, and 2 oranges and 7 apples are worth 38 dollars. How much are 4 oranges and 2 apples worth, in dollars?

26. You are one of six people who need to sit in a row of six chairs. The chairs are arranged horizontally. The following conditions apply:

    - Michael and Aditya talk a lot and cannot sit next to each other.
    - You cannot sit right next to Michael or Jake.
    - Rebecca and Sally must sit with each other.
    - You do not sit in the last chair (rightmost chair).
    - Michael loves to sit in the first chair.
    - You are not friends with Sally so you do not sit with her.

    If Rebecca sits in the 3rd chair, who sits in the 6th chair in the row? (for this question, put 1 if it is Michael, 2 if it is Aditya, 3 if it is Jake, and 4 if it is Sally, if you put a name instead of the corresponding number you will receive NO credit.)

27. A four team tournament has each team play every other team twice. A team earns 4 points for a win, 3 for a tie, and 1 for a loss. If the teams earn 67 points in total, how many ties occurred?

28. What is the remainder when $5^0 + 5^1 + 5^2 + ... + 5^{2023}$ is divided by 7?

29. Suppose we have a perfect square A. When we add 45 to A, we get another perfect square which we can call B. When we add 45 to B, we get a number that is 2 less than a perfect square. What is the square root of B?

30. Consider a triangle $\triangle ABC$ with $AB = 13, BC = 14, CA = 15$. Let D be the foot of the altitude from A to BC. Let E be the point on AC such that $\angle EDC = \angle CAD$, and let F be the point on AB such that $\angle FDB = \angle BAD$. If the length of EF can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

3 Solutions

1. This is just $3^5 = \boxed{243}$.
2. Since the area of a triangle is $\frac{1}{2}$ times base times height, the answer is $\boxed{6}$.
3. Since the mean of the first 5 numbers is 20, their sum is 100. Thus, x is 58 and the median of the numbers is $\boxed{12}$.
4. There are 2 choices and 52 cards so the answer is $\frac{2}{52} = \frac{1}{26}$ which is expressed as $\boxed{27}$.
5. This is just 5 choose 2 which is $\boxed{10}$.
6. $\boxed{42}$
7. This is just $\frac{1}{4}$ so the answer is $\boxed{5}$.
8. Bash out all possible products to find that only $18 * 8 = 144$ is a perfect square. Thus, the answer is $\boxed{1}$.
9. Two cases are possible. Either Bob, Leroy, and Sam are pirates and Calvin is a wizard or Bob, Leroy, and Calvin are pirates and Sam is a wizard. Either way, the number of wizards is $\boxed{1}$.
10. The initial total volume of the prism is 120 inches cubed. The holes that are cut into each side have volumes of 36, 45, and 54 inches cubed respectively, leaving the prism with a volume of $120 - 36 - 45 - 54 = -15$ inches cubed. However, in subtracting these volumes, you subtract the volume of the 3 by 3 by 3 cube at the center of the prism twice. Thus, you add back $2 \times 27 = 54$. $-15 + 54 = \boxed{39}$.
11. $\frac{1}{2}^5 = \frac{1}{32}$, so our answer is $1 + 32 = \boxed{33}$
12. Through the definition of the palindrome, we get the following forms for n-digit palindromes.

$$\begin{array}{lcl} 1 \text{ digit} & - & A \\ 2 \text{ digit} & - & AA \\ 3 \text{ digit} & - & ABA \\ 4 \text{ digit} & - & ABBA \\ & \cdots & \end{array}$$

The one digit case is a special case because it is the only one where the number of possibilities for the first digit is 10 (since 0 is a valid one-digit number). Otherwise, by working through the obvious combinatorics, we get the following:

$$\begin{array}{lcl} 1 \text{ digit} & - & 10 \\ 2 \text{ digit} & - & 9 \\ 3 \text{ digit} & - & 9 * 10 \\ 4 \text{ digit} & - & 9 * 10 \\ 5 \text{ digit} & - & 9 * 10^2 \\ 6 \text{ digit} & - & 9 * 10^2 \\ & \cdots & \end{array}$$

Now the problem comes down to simply summing all these up to the 10 digit case.

$$\begin{aligned}
10 + 9 + 2(9 \times 10) + 2(9 \times 10^2) + 2(9 \times 10^3) + 2(9 \times 10^4) \\
= 19 + 18(10 + 10^2 + 10^3 + 10^4) \\
= 19 + 18(11110) \\
= 19 + 2(99990) \\
= 19 + 199,980 \\
= 199,999 \\
= 2 \times 10^6 - 1
\end{aligned}$$

Giving us $a = 2$, $n = 6$, and $k = 1$.
$2 + 6 + 1 = \boxed{9}$

13. The letter C is at the start twice in every five orderings. In the first 2020 orderings, there are 2020 * 2/5 = 808 such orderings. In the last three, there is one additional ordering, resulting in an answer of $\boxed{809}$.

14. By Vieta's, you get that the roots must be -1 and -2023, -7 and -289, or -17 and -119. Since a is the sum of the roots, the smallest sum is $-17 - 119 = -136$. Thus, the answer is $\boxed{136}$.

15. Roots: $-5, 5$

    $-5 \times 5 = -25$
    $-25^2 = \boxed{625}$

16. Logarithmic functions grow slower than linear ones except at small values of x; as such, we only need to find 2 intersection points to see the range of values that satisfy the inequality. $log_2 x$ and $0.5x$ intersect at $x = 2$ and $x = 4$: as such, the $\boxed{3}$ integers 2, 3, and 4 satisfy the inequality.

17. Note that all 4 times the dealer goes through all the cards, he flips through the card labeled 5 and every 12th card after.

    An arithmetic sequence can be derived from this knowledge, starting from 5 and counting every 12th number after until 2021 (i.e. $5, 17, 29...1229$).

    From this arithmetic sequence, groups of 12 can be isolated in which the same number of cards will end face up as a set that begins with a starting term and ending before each successive term (i.e. $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$).

    Now, we can count how many cards end face up in the first set, and we find that $6, 9, 10, 11, 12, 13,$ and 16 are flipped.

    Each set therefore has 7 cards that are left face up in each set, and there are $\frac{1229-5}{12} = 102$ such sets, and $7 \cdot 102 = 714$ cards.

To account for the cards that are not considered, we know that for the cards $1, 2, 3$, and 4, each card is flipped once in total, which means that they all end face up. This adds 4 cards to our total.

$1229, 1230, 1231, 1232, 1233$, and 1234 are also not included in any set from the previous, and since each number in the previous arithmetic sequence is flipped all 4 times, all flips coincide on the number 1229.

We can calculate the numbers that have an odd number of flips, and we obtain 1230, 1233, and 1234. This adds 3 additional cards.

Now we can sum up all the cards that will be left face up: $714 + 4 + 3 = 721$ cards.

The total number of cards that end face up is $\boxed{721}$.

18. As one can see, in the equation, $x^2y - x^2 - 3y + 4 = 14$ the naive approach would be to move the 14 to the left side and solve for the zeros of the equation. However, this is pretty hard to factorize and solve.

    Instead, x and y being positive integers should give us a hint to factorize the left hand side such that we can solve for all the factors of the LHS by using the factors of the single integer on the right hand side.

    First, we rewrite the equation as follows to make it easier for us to factorize.

    $x^2y - x^2 - 3y + 3 = 13$

    Then, we can rearrange this equation to become $x^2(y - 1) - 3(y - 1) = 13$

    This allows us to easily factorize the left hand side to become $(y - 1)(x^2 - 3) = 13$

    Because 13 is a prime number, one of the factors on the left hand side must be equal to 1. This is because x and y are positive integers, so one can observe that the factors in the LHS must also be integers.

    There are two possibilities:
    1. $y - 1$ is 1, so y=2 and x=4
    2. $x^2 - 3$ is 1, so x=2 and y=14

    The sum of the products of these solutions is $\boxed{36}$.

19. The absolute value of this sum is simply the sum of all integers between 1 and 2023. Using the formula for an arithmetic series, the sum is $\frac{2023*2024}{2} = 2047276 \equiv \boxed{276} \pmod{1000}$.

20. For this problem, since there are 5 letters in MCAMC, if all of these letters were distinct, there would be 5! (factorial), or 120 different possibilities. However, the two 'M' characters are interchangeable and the two 'C' characters are also interchangeable, so we must divide 120 by 2×2, or 4, getting us the answer $\boxed{30}$

21. DEF is the medial triangle of ABC meaning that is has one fourth the area of ABC. The area of ABC is $4\sqrt{3}$. One fourth of that is $\sqrt{3}$. The square of that is $\boxed{3}$.

22. The prime factorization of 120 is $2^3 * 3 * 5$. As such, $x = -2 * 3 * 5 = -30$ in order to give each prime factor an even power. $y = 3^2 * 5^2 = 225$ so each exponent is a multiple of 3. $|-30 + 225| = \boxed{195}$.

23. We can convert both the equations into $2logx = a$ and $blogx = a^2b$ using the properties of logs. Dividing both equations yields $\frac{b}{2} = ba$. Dividing both sides by b gives us that $a = \frac{1}{2}$ meaning that the answer is $\boxed{3}$.

24. The sum of the first n positive odd integers is n^2. If $1000n$ is divisible by n^2, then 1000 must be divisible by n. The number of positive integer factors of 1000 is $\boxed{16}$.

25. This can be written as the following system of equations:

$$x + 3y = 17$$

$$2x + 7y = 38$$

Solving using elimination, we get $x = 5$ and $y = 4$, making $4x + 2y = \boxed{28}$.

26. Tracing the possibilities, we see that Jake is the only person who could be sitting in the chair, so $\boxed{3}$ is our answer (answers of "Jake" received NO credit).

27. Because 67 is the combined score for all teams, we can simply consider each tie as adding 6 points and each match with a result as adding 5 points. The number of matches in a double round robin 4 team tournament is $2 * \binom{4}{2} = 12$. Using the equations $x + y = 12$ and $5x + 6y = 67$, we find that y, representing the number of ties, equals $\boxed{7}$.

28. The remainder when powers of 5 are divided by 7, starting from the first power, follow a pattern: $5, 4, 6, 2, 3, 1$. Thus, there is a cycle of 6. Since the sum of the remainders in every cycle is 21, which is a multiple of 7, we can ignore their contribution to the sum. Since 2023 mod 6 is 1, we know that the last term is 5 mod 7. Furthermore, we ignored the first term, 5^0, which is 1 mod 7. Thus, the total sum of the remainders is just $\boxed{6}$.

29. $A = 484$ and $B = 529$. Thus, the square root of $B = \boxed{23}$.

30. First, from Heron's Formula, we find that

$$[ABC] = \sqrt{21(8)(7)(6)} = \sqrt{7^2 \cdot 12^2} = 84.$$

The area is also $BC \cdot AD/2$, and so we find

$$AD = \frac{168}{14} = 12.$$

From the Pythagorean Theorem, we also find $BD = 5$ and $CD = 9$.

Now, notice that $\angle BAD = 90 - \angle B = \angle BDF$, so we conclude that $\angle BFD = 90$. Similarly, $\angle CED = 90$. Using similar methods to finding AD, we conclude that $DF = \frac{60}{13}$ and $DE = \frac{36}{5}$. From the Pythagorean Theorem, we see that $AF = \frac{144}{13}$ and $AE = \frac{48}{5}$.

Now, from Ptolemy's Theorem, we find that

$$AD \cdot EF = AF \cdot DE + AE \cdot DF \to 12 \cdot EF = \frac{144 \cdot 36 + 48 \cdot 60}{65}.$$

Solving for EF yields

$$EF = \frac{672}{65},$$

for an answer of $672 + 65 = \boxed{737}$.